08/08/2018

MODULE - 1

1

INFORMATION THEORY

Measure of Information:-

An information source generates the information, however the amount of information conveyed by the source is different for different symbols it emits. A mathematical model can be formulated to quantify the amount of information conveyed by a ~~signal~~. symbols.

Consider the following statements:

1. It rained heavily in Agunde yesterday.
2. There was heavy rainfall in Rajasthan last Night.

Although Both statements infer the occurence of rainfall, but the amount of information conveyed by them is different.

The 1st statement is a common event, the 2nd statement will create some degree of surprise.

It is clear that the amount of information conveyed by a message is inversely proportional to the occurence.

Therefore the amount of information or self information or message K is given by

$$I_K \propto \frac{1}{P_K} \qquad \boxed{I_K \propto \frac{1}{P_K}} \qquad \underline{I_K \propto \frac{1}{P_K}}$$

$I_K \propto$

PROPERTIES:-

1) The information conveyed by message cannot be negative (-ve)

$$I_K \geq 0$$

2) If the event is definite, that is, the probability of occurence of event is '1', then the information conveyed will be '0',

$$P_K = 1$$

$$\Rightarrow I_K = 0$$

3) The information conveyed by composite statement, which is independent is given by the sum of individual self information.

$$I(m_1, m_2) = I(m_1) + I(m_2)$$

4) The only mathematical operator which satisfies the above relations is logarithmic operator.

$$I_K = \log_r \left(\frac{1}{P_K}\right) \text{ units}$$

$$I_K = -\log_r (P_K) \text{ units}$$

a] If $r = 2$, unit is bits

b] If $r = e$, unit is nats

c] If $r = 10$, unit is Hardley (or) Decits

PROBLEMS :-

1] Consider a Binary system emitting 2 symbols 0, 1 with ~~properties~~ probabilities 0.6 and 0.4. Find the information conveyed by bit 0 and bit 1.

Ans:- $I_K = \log_r \left(\frac{1}{P_K}\right)$ units

$$I_K = \log_2 \left(\frac{1}{P_K}\right) \underset{\text{bits}}{\text{units}}$$

$$I_0 = \log_2 \left(\frac{1}{0.6}\right) \text{ bits} = 0.7370 \text{ bits}$$

$$I_1 = \log_2 \left(\frac{1}{0.4}\right) \text{ bits} = 1.3219 \text{ bits}$$

2] Consider a source emitting 2 symbols. S_0 and S_1 with corresponding ~~propert~~ probabilities $3/4$ and $1/4$. Find the self information of the symbols in (i) bits 2

(ii) Hartley 10

(iii) nats e

1) $I_K = \log_r \left(\frac{1}{P_K}\right)$ units

a] Bits

$$I_{S_0} = \log_2 \left(\frac{1}{3/4}\right) \text{ bits} = 0.4150 \text{ bits}$$

$$I_{S_1} = \log_2 \left(\frac{1}{1/4}\right) \text{ bits} = 2 \text{ bits}$$

b] Hartley

$$I_{S_0} = \log_{10} \left(\frac{1}{3/4}\right) \text{ hartley} = 0.1249 \text{ Hartley}$$

$$I_{S_1} = \log_{10} \left(\frac{1}{1/4}\right) \text{ Hartley} = 0.6021 \text{ Hartley}$$

c] nats

$$I_{S_0} = \log_e \left(\frac{1}{3/4}\right) \text{ nats} = 0.2877 \text{ nats}$$

$$I_{S_1} = \log_e \left(1/(1/4)\right) \text{ nats} = 1.3863 \text{ nats}$$

3] A pair of dice is rolled simultaneously, the outcome of the 1st one is considered to be X and 2nd one as Y. 3 events are identified

(i) $P = \{(X, Y) : (X + Y)$ is exactly divisible by 3

(ii) $Q = \{(X, Y) : (X + Y)$ is an even number

(iii) $R = \{(X, Y) : (X + Y)$ is $7\}$.

Find the probability and the information conveyed by each one.

Ans:- $S = \{ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6)$

$(2,1), (2,2), (2,3), (2,4), (2,5), (2,6)$

$(3,1), (3,2), (3,3), (3,4), (3,5), (3,6)$

$(4,1), (4,2), (4,3), (4,4), (4,5), (4,6)$

$(5,1), (5,2), (5,3), (5,4), (5,5), (5,6)$
$(6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \}$

(i) $P = \{ (1,2), (1,5), (2,1) (2,4), (3,3), (3,6), (4,2), (4,5), (5,1) (5,4), (6,3), (6,6) \}$

$P_P = \frac{12}{36} = \frac{1}{3}$

$I_P = \log_r \left(\frac{1}{P_K}\right)$ units

$I_P = \log_2 \left(\frac{1}{1/3}\right)$ bits $= 1.5850$ bits

(ii) $Q = \{ (1,1), (1,3), (1,5), (2,2) (2,4), (2,6), (3,1), (3,3), (3,5), (4,2), (4,4), (4,6), (5,1), (5,3), (5,5), (6,2), (6,4), (6,6) \}$

$P_Q = \frac{18}{36} = \frac{1}{2}$ ×

$I_Q = \log_2 \left(\frac{1}{1/2}\right) = \log_2 (2) = 1$ bit

(iii) $R = \{ (1,6), (2,5), (3,4), (4,3), (5,2), (6,1) \}$

$P_R = \frac{6}{36} = \frac{1}{6}$

$I_R = \log_2 \left(\frac{1}{1/6}\right) = 2.5850$ bits

09/05/2018

AVERAGE INFORMATION CONTENT (ENTROPY) OF ~~INDEPE~~ SYMBOLS IN LONG INDEPENDENT SEQUENCES

(ZERO MEMORY SOURCE)

Consider a Zero Memory Source in which emission of the current symbol is not dependent on emission of previous symbol.

Consider a source emitting symbols

$$S = \{s_1, s_2, \ldots \ldots s_N\}$$

with respective probabilities

$$P = \{p_1, p_2, \ldots \ldots \ldots p_N\}$$

In a long message containing L symbols, S_1 will occur on an average $p_1 L$. Similarly S_2 will appear $p_2 L$ times and so on. S_N appears $p_N L$ times.

The self information of symbol S_1 in each symbol is

$$\boxed{I_{S_1} = \log_2\left(\frac{1}{P_1}\right)} \text{ bits}$$

Therefore $P_1 L$ number of symbols are present on an average in a length of L symbols.

Therefore the total information conveyed by symbols of type S_1

Average Information

$$= p_1 L \log_2\left(\frac{1}{p_1}\right)$$

Similarly for S_2, it is $p_2 L \log_2\left(\frac{1}{p_2}\right)$

for S_N ; $= p_N L \log_2\left(\frac{1}{p_N}\right)$

The total information will be

$$I_{Total} = p_1 L \log_2\left(\frac{1}{p_1}\right) + p_2 L \log_2\left(\frac{1}{p_2}\right) + \cdots \cdots p_N L \log_2\left(\frac{1}{p_N}\right).$$

$$I_{total} = L \sum_{i=1}^{N} p_i \log_2\left(\frac{1}{p_i}\right)$$

The average information conveyed by the source emitting L symbols is called as entropy denoted by $H(s)$

$$H(s) = \frac{I_{total}}{L}$$

$$\boxed{H(s) = \sum_{i=1}^{N} p_i \log_2\left(\frac{1}{p_i}\right)} \text{ bits/symbol}$$

$H(s)$ is the average uncertainity or average amount of surprise.

The average information rate R_s is defined as

$$\boxed{R_s = r_s H(s)} \text{ bits/sec (or) BPS}$$

UNIT

r_s = symbol rate / baud rate — symbols/sec

(1) A discrete memoryless source emits 5 symbols in 2 ms. The symbol probabilities are $\{0.5, 0.25, 0.125, 0.0625, 0.0625\}$. Find the entropy and the average information rate of the source.

Ans:-

$$H(s) = \sum_{i=1}^{N} p_i \log_2\left(\frac{1}{p_i}\right) \text{ bits/symbol}$$

$$H(s) = \sum_{i=1}^{5} p_i \log_2\left(\frac{1}{p_i}\right)$$

$$H(s) = 0.5 \log_2\left(\frac{1}{0.5}\right) + 0.25 \log_2\left(\frac{1}{0.25}\right) + 0.125 \log_2\left(\frac{1}{0.125}\right) + 0.0625 \log_2\left(\frac{1}{0.0625}\right) + 0.0625 \log_2\left(\frac{1}{0.0625}\right)$$

$H(s) = 0.5 + 0.5 + 0.3750 + 0.25 + 0.25$

$H(s) = 1.8750$ bits/symbol

Average information rate of the source

$R_s = r_s \, H(s)$

$r_s = \frac{5 \text{ symbols}}{2 \text{ ms}} = \frac{5}{2 \times 10^{-3}} = 2500$ symbols/sec

$R_s = 2500 \times 1.8750$

$R_s = 4687.5$ bits/sec ≈ 4.687 Kbits/sec

(2) The output of an information source contains 160 symbols, 128 of which occur with probability of $1/256$ and remaining with probability of $1/64$ each. Find the average information content and average information rate of the source if the source emits 10,000 symbols/sec.

Ans:- $H(s) = \sum_{i=1}^{N} p_i \log_2 \left(\frac{1}{p_i}\right)$ bits/symbol

$r_s = 10,000$ symbols/sec.

128 symbols —— $1/256$ $\qquad N = 160.$

32 symbols —— $1/64$

$H(s) = 128 * \left(\frac{1}{256}\right) \log_2\left(\frac{1}{1/256}\right) + 32 * \left(\frac{1}{64}\right) \log_2\left(\frac{1}{1/64}\right)$

$H(s) = 7$ bits/symbol

Average rate of source.

$R(s) = r_s \, H(s)$

$= 10,000 * 7$

$R(s) = 70,000$ bits/sec ≈ 70 kbit/sec.

③ A discrete source emits one of the 5 symbols in every second. The symbol probabilities are 1/4, 1/8, 1/8, 3/16, 5/16. Find the average information content of the source in Hartley/symbol & nats/symbol.

Ans
$$H(s) = \sum_{i=1}^{N} p_i \log_r\left(\frac{1}{p_i}\right)$$

(i) Hartley/symbol

$$H(s) = \sum_{i=1}^{N} p_i \log_{10}\left(\frac{1}{p_i}\right)$$

$$H(s) = \frac{1}{4}\log_{10}\left(\frac{1}{1/4}\right) + \frac{1}{8}\log_{10}\frac{1}{(1/8)} + \frac{1}{8}\log_{10}\frac{1}{(1/8)}$$

$$+ \frac{3}{16}\log_{10}\frac{1}{(3/16)} + \frac{5}{16}\log_{10}\frac{1}{(5/16)}$$

$$\boxed{H(s) = 0.6705 \text{ Hartley/symbol}}$$

(ii) nats/symbol

$$H(s) = \sum_{i=1}^{N} p_i \log_e\left(\frac{1}{p_i}\right)$$

$$H(s) = \frac{1}{4}\log_e\frac{1}{(1/4)} + \frac{1}{8}\log_e\frac{1}{(1/8)} + \frac{1}{8}\log_e\frac{1}{(1/8)} +$$

$$\frac{3}{16}\log_e\frac{1}{(3/16)} + \frac{5}{16}\log_e\frac{1}{(5/16)}$$

$$\boxed{H(s) = 1.5438 \text{ nats/symbol.}}$$

④ Consider a system emitting 3 symbols X, Y, Z with probabilities 0.5, 0.3, 0.2 respectively. Find the information conveyed by each of these symbols.

Ans:- ~~E(s)=~~ $I_X = \log_2\left(\frac{1}{0.5}\right) = 1 \text{ bit}$ $\quad I_Z = \log_2\left(\frac{1}{0.2}\right) = 2.3219 \text{ bits}$

$$I_Y = \log_2\left(\frac{1}{0.3}\right) = 1.7370 \text{ bits}$$

$I_{total} = I_x + I_y + I_z$

$I_{total} = \underline{5.0589}$ bits

(5) A source emits 6 symbols which have the probabilities 0.3, 0.22, 0.2, 0.12, 0.1, 0.06. Find the entropy.

Ans

$$H(S) = \sum_{i=1}^{N} p_i \log_2\left(\frac{1}{p_i}\right)$$

$$H(S) = 0.3 \log_2\left(\frac{1}{0.3}\right) + 0.22 \log_2\left(\frac{1}{0.22}\right) + 0.2 \log_2\left(\frac{1}{0.2}\right) +$$

$$0.12 \log_2\left(\frac{1}{0.12}\right) + 0.1 \log_2\left(\frac{1}{0.1}\right) + 0.06 \log_2\left(\frac{1}{0.06}\right)$$

$H(S) = \underline{2.4088}$ bits/symbol.

***

(6) Consider a binary source emitting 2 symbols X and Y. Let the probability of emission of X be p. Plot the function of $H(S)$ versus p.

Ans:- ~~H(S) =~~ $P_x = p \qquad P_y = 1-p$

$$H(s) = \sum_{i=1}^{N} P_i \log_2\left(\frac{1}{P_i}\right)$$

Since there are 2 symbols X and Y

$$P_x + P_y = 1$$

given that

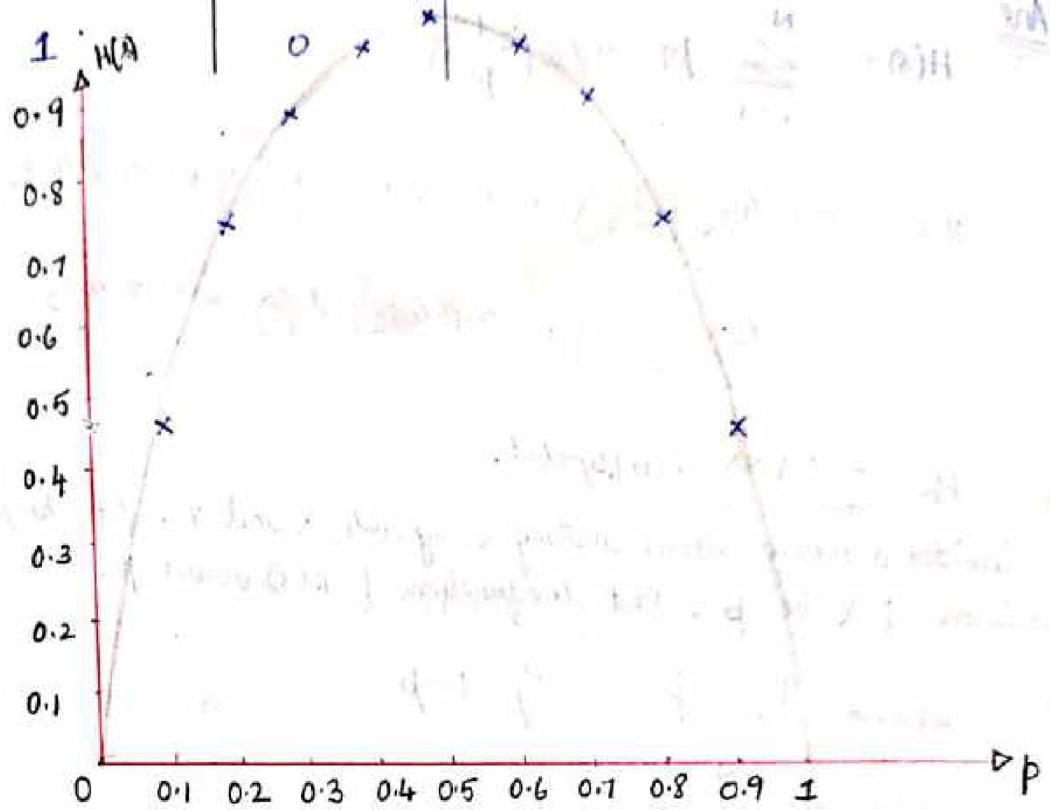
$$P_x = p$$

$$\therefore P_y = 1-p$$

$$H(s) = p \log_2\left(\frac{1}{p}\right) + (1-p) \log_2 \frac{1}{(1-p)}$$

p	$H(s)$
0.1	0.4690
0.2	0.7219
0.3	0.8813
0.4	0.9710

0.5	1
0.6	0.9710
0.7	0.8813
0.8	0.7219
0.9	0.4690
1	0



Since it is a Binary channel, the maximum value is 1 at $p = 0.5$

(7) The International Morse code uses sequence of symbols of dots and dashes to transmit letters of english alphabet. The dash is represented by a pulse of duration 2ms and dot by duration 1 ms. The probability of dash is 1/2 as that of dot. Consider a 1 ms duration of gap between the symbols

(i) Calculate the self information of dot and dash

(ii) The average information content of dot and dash code.

(iii) Average rate of information.

Ans:- $p_{dash} + p_{dot} = 1$

$p_{dash} = \frac{1}{2} p_{dot}$

⑥

$$\frac{1}{2} p_{dot} + p_{dot} = 1$$

$$p_{dot} = \frac{2}{3} \qquad \frac{3}{2} = 1$$

$$p_{dot} = \underline{\underline{0.6667}}$$

$$p_{dash} = \underline{\underline{0.3333}}$$

$$H(s) = \sum_{i=1}^{N} p_i \log_2\left(\frac{1}{p_i}\right)$$

$$H(s) = 0.6667 \log_2\left(\frac{1}{0.6667}\right) + 0.3333 \log_2\left(\frac{1}{0.3333}\right)$$

(ii) $H(s) = 0.9183$. bits/symbol

(ii) $I_{dash} = \log_2\left(\frac{1}{1/3}\right) = 1.5850$ bits

$I_{dot} = \log_2\left(\frac{1}{2/3}\right) = 0.5850$ bits

$p_{dot} = 2/3$

(iii) $R_s = r_s H(s)$ $\qquad p_{dash} = 1/3$

$\cdot \quad \cdot \quad —$

$1 \; 1 \; 1 \; 1 \; 2 \; 1 = 3$ symbols

$$\boxed{\cdot \qquad \cdot \qquad —}$$

$\cancel{H(s) =}$ $\cdot \; 1 + 1 + 1 + 1 + 2 + 1 = \underline{\underline{7\,ms}}$

$$r_s = \frac{3 \text{ symbol}}{7 \text{ m sec}}$$

$$r_s = \frac{3000}{7}$$

$$R_s = \frac{3000}{7} * 0.9183$$

$R_s = 131.1857^{*3}$ bits/sec.

$R_s = \underline{\underline{393.55}}$ bits/sec

$$\boxed{R_s = 393.55 \text{ bits/sec}}$$

(8) Same as previous

A dash is represented by 3 ms and a dot by 1 ms. The probability of dash is $\frac{1}{3}$ as that of dot.

Ans: $p_{dot} + p_{dash} = 1$

$$p_{dash} = \frac{1}{3} p_{dot}$$

$$p_{dot} + \frac{1}{3} p_{dot} = 1$$

$$p_{dot} = \frac{3}{4}$$

$$p_{dash} = \frac{1}{4}$$

$$I_{dash} = \log_2 \left(\frac{1}{1/4} \right) = 2 \text{ bits}$$

$$I_{dot} = \log_2 \left(\frac{1}{3/4} \right) = 0.4150 \text{ bits}$$

$$H(S) = \sum_{i=1}^{N} p_i \log_2 \left(\frac{1}{p_i} \right)$$

$$H(S) = \frac{3}{4} \log_2 \left(\frac{4}{3} \right) + \frac{1}{4} \log_2 (4)$$

$$H(S) = 0.8113 \text{ bits/symbol}$$

$\cdot \quad \cdot \quad \cdot \quad —$

$1 + 1 + 1 + 1 + 1 + 1 + 3 + 1 = 10\text{ ms}$

$$r_s = \frac{4}{10\text{ms}} = 400 \text{ ~~bits/sec~~ symbols/sec}$$

$R_s = r_s H(s)$

$R_s = 324.52$ bits/sec

PROPERTIES OF ENTROPY :-

1] Entropy is a continuous function of probability, that is as p_i varies from 0 to 1, so does the entropy function. (for proof refer problem 6).

2] Entropy is a symmetric function of its arguments, that is, Entropy of a system is same irrespective of the order in which the symbols are arranged.

Eg:- $S_A = \left\{\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right\}$.

$$H_A(s) = \frac{1}{4}\log_2(4) + \frac{1}{2}\log_2(2) + \frac{1}{4}\log_2(4)$$

$$H_A(s) = 1.5 \text{ bits/symbol}$$

$$S_B = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right\}$$

$$H_B(s) = \frac{1}{2}\log_2(2) + \frac{1}{4}\log_2(4) + \frac{1}{4}\log_2(4)$$

$$H_B(s) = 1.5 \text{ bits/symbol}$$

$\Rightarrow H_A(s) = H_B(s)$

***3] Upper Bound on entry [EXTREMAL PROPERTY]

Eg:- Consider election results yet to be announced

Case(i) 2 equally strong candidates

Case(ii) 1 contestant is very strong and the other one is least expected to win.

Case (i) gives maximum information, that is entropy of a source is maximum for equi-probable symbols.

Proof:

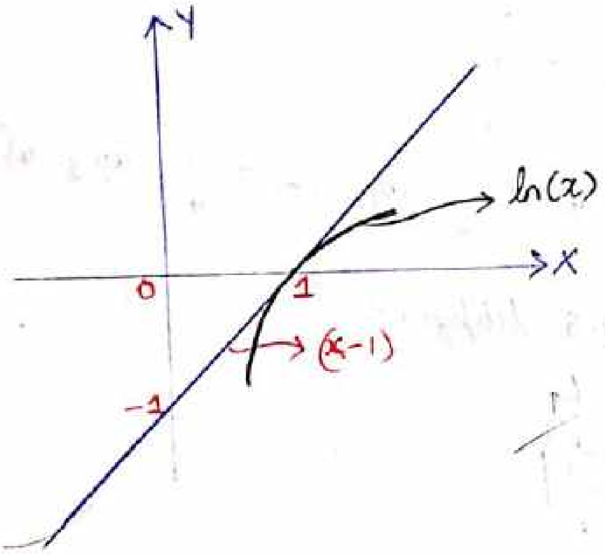
Consider $\log_2 N - H(s)$

$$= \log_2 N - \sum_{i=1}^{N} p_i \log_2\left(\frac{1}{p_i}\right) \qquad \because \sum_{i=1}^{N} p_i = 1$$

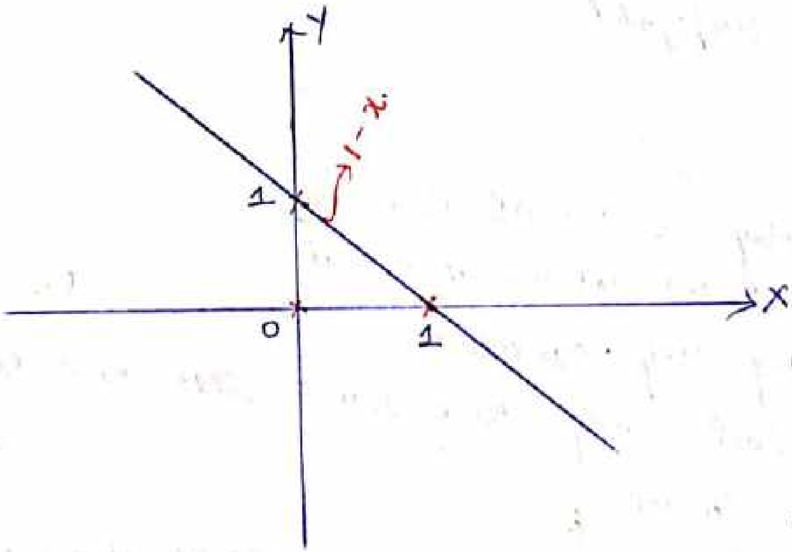
$$= \sum_{i=1}^{N} p_i \log_2 N - \sum_{i=1}^{N} p_i \log_2\left(\frac{1}{p_i}\right)$$

$$\log_2 N - H(s) = \sum_{i=1}^{N} p_i \log_2 (N p_i) \quad \text{—(1)}$$

Consider $\boxed{\ln(x) \le x-1}$



$$-\ln(x) \ge 1 - x \quad \text{—(2)} \qquad \text{or} \qquad \ln\left(\frac{1}{x}\right) \ge (1-x) \quad \text{—(2)}$$



Let $x = \frac{1}{N p_i}$

Using equation (2) in (1)

$$\log_2 N - H(s) = \sum_{i=1}^{N} p_i \frac{\log_e (Np_i)}{\log_e 2}$$

$$\log_2 N - H(s) = \log_2 e \sum_{i=1}^{N} p_i \log_e (Np_i)$$

$$\log_2 N - H(s) \geq \log_2 e \sum_{i=1}^{N} p_i \left[1 - \frac{1}{Np_i}\right]$$

$$\log_2 N - H(s) \geq \log_2 e \left[\sum_{i=1}^{N} p_i - \sum_{i=1}^{N} \frac{1 \cancel{p_i}}{N \cancel{p_i}}\right]$$

$$\log_2 N - H(s) \geq \log_2 e \left[1 - 1\right]$$

$$\boxed{\log_2 N - H(s) \geq 0}$$

$$\boxed{H(s) \leq \log_2 N}$$

$\because \frac{1}{N} \sum_{i=1}^{N} 1 = \frac{1}{N}[N-1+1] = 1$

Equiprobable

$H(s)_{max} = \log_2 N$

From the above equation we have upper Bound $H(s)_{max} = \log_2 N$

From the graph we have the equality symbol holds only when $x = 1$,

that is $x = 1 = \frac{1}{Np_i}$

$$\boxed{p_i = \frac{1}{N}}$$

From the above equation, it is clear that for maximum entropy all the symbols should be equi-probable.

4] The lower Bound entropy is zero.

16/08/18

5] Property of additivity :-

Assume we have symbols $S_1, S_2, \ldots \ldots S_N$ with corresponding propabilities $P_1, P_2, \ldots \ldots P_N$.

Suppose we split the symbol S_N into m symbols, such that

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$S_N = S_{N_1}, S_{N_2}, \ldots\ldots S_{N_m}$ with probabilities

$P_N = P_{N_1}, P_{N_2}, \ldots\ldots P_{N_m}$ respectively.

We have $P_N = P_{N_1} + P_{N_2} + \ldots\ldots P_{N_m} = \sum_{j=1}^{m} P_{N_j}$ — (1)

$$H'(s) = H(P_1, P_2, P_3, \ldots\ldots P_{N-1}, P_{N_1}, P_{N_2}, P_{N_3} \ldots\ldots P_{N_m})$$

$$H'(s) = \sum_{i=1}^{N-1} P_i \log_2\left(\frac{1}{P_i}\right) + \sum_{j=1}^{m} P_{N_j} \log_2\left(\frac{1}{P_{N_j}}\right)$$

$$H'(s) = \sum_{i=1}^{N} P_i \log_2\left(\frac{1}{P_i}\right) - \sum P_{N_j} \log_2\left(\frac{1}{P_N}\right) + \sum_{j=1}^{m} P_{N_j} \log_2\left(\frac{1}{P_{N_j}}\right)$$

$$H'(s) = H(s) - \sum_{j=1}^{m} P_{N_j} \log_2\left(\frac{1}{P_N}\right) + \sum_{j=1}^{m} P_{N_j} \log_2\left(\frac{1}{P_{N_j}}\right)$$

$$\boxed{\cancel{H'(s) = H(s)}}$$

$$= H(s) + \sum_{j=1}^{m} P_{N_j} \log_2\left(\frac{1}{P_{N_j}}\right) - \sum_{j=1}^{m} P_{N_j} \log_2\left(\frac{1}{P_N}\right)$$

$$= H(s) + \sum_{j=1}^{m} P_{N_j} \log_2\left(\frac{P_N}{P_{N_j}}\right)$$

Since Always $P_{N_j} \leq P_N$

The 2nd term in the RHS will be a positive quantity.

$H'(s) = H(s) +$ Positive quantity.

$$\boxed{H'(s) \geq H(s)}$$

Partitioning the symbols into subsymbols, the total entropy does not decrease.

16/08/2018

SOURCE EFFICIENCY & REDUNDANCY

The source efficiency is defined as the ratio of average information conveyed to the maximum average information.

$$\eta_s = \frac{H(s)}{H(s)_{max}} \times 100\%$$

Source redundancy is

$$R_{\eta_s} = 1 - \eta_s$$

$$\boxed{R_{\eta_s} = 100 - \eta_s}$$

$$= \frac{H(s)_{max} - H(s)}{H(s)_{max}} \times 100\%$$

9. Consider a source emitting 3 symbols A, B and C with probabilities 0.7, 0.15, 0.15 respectively. Calculate the source efficiency & Redundancy.

Ans:- $H(s) = \sum_{i=1}^{N} p_i \log_2\left(\frac{1}{p_i}\right)$

$$H(s) = 0.7 \log_2\left(\frac{1}{0.7}\right) + 0.15 \log_2\left(\frac{1}{0.15}\right) + 0.15 \log_2\left(\frac{1}{0.15}\right)$$

$$H(s) = 1.1813 \text{ BPS}$$

$$H(s)_{max} = \log_2 N$$

$$\boxed{N = \frac{1}{p_i}}$$

$$\boxed{H(s)_{max} = 1.5850}$$

$$\eta_s = \frac{H(s)}{H(s)_{max}} \times 100\%$$

$$\eta_s = \frac{1.1813}{1.5850} \times 100\%$$

$$\boxed{\eta_s = 74.53\%}$$

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$$R_{\eta_s} = 1 - \eta_s$$

$$= 100 - 74.53$$

$$\boxed{R_{\eta_s} = \underline{25.47\,\%}}$$

(10) A certain data source has 8 symbols that are reduced in blocks of 4 at the rate of 500 blocks per second. The first symbol in each block is always the same (used for synchronisation). The remaining 3 are filled by any 1 of the 8 symbols with equal probability. What is the entropy rate of the source.

Ans:- Given:

The first symbol is used for synchronisation & is repetitive in every block. Hence the amount of information equals zero.

Synchronisation			
H_1	H_2	H_3	H_4

No. of symbols = 8

$H_1 = 0$ [Since 1st symbol is always known. So amount of information $I_i = 0$ $\Rightarrow I_i = \log_2\left(\frac{1}{p_i}\right)$

$$\Rightarrow H(s) = \sum_{i=1}^{N} p_i I_i$$

$$= \underline{0}\ ]$$

$H_2 = \log_2 N = \log_2 8 = 3$ BP~~blocks~~ symbol

$H_3 = \log_2 N = \log_2 8 = 3$ Bits per ~~Blocks~~ symbol

$H_4 = \log_2 N = \log_2 8 = 3$ Bits per ~~Blocks~~ Symbol

Equi Probable

$H(s) = \log_2 N$

$H_{total} = H_1 + H_2 + H_3 + H_4$ ✓

$$\boxed{H_{total} = 9 \text{ bits per blocks}}$$

Entropy rate

$R_s = r_s \underbrace{H(s)}$

$R_s = \text{blocks/sec} * \text{bits/blocks}$

$R_s = \text{bits/sec}$

$r_s = 500$

$R_s = 500 * 9$

$\boxed{R_s = \underline{\underline{4500}} \text{ bits/sec}}$

(11.) A certain digital frame consists of 15 fields. The 1st and the last field of each frame are same. The remaining 13 fields can be filled by any 1 of the 16 symbols with equal probability. Find the average information conveyed by the frame. Also find the average rate of information if 100 frames are transmitted every second.

Ans:-

H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8	H_9	H_{10}	H_{11}	H_{12}	H_{13}	H_{14}	H_{15}
0														0

$H_1 = 0$

$H_{15} = 0$

$H_2 = H_3 = H_4 = \ldots H_{14} = \log_2 N$

$= \log_2(16)$

$= \underline{\underline{4}}$ bits/symbol OR Bits/fields.

$H_{total} = H_1 + H_2 + H_3 + H_4 + H_5 + H_6 + H_7 + H_8 + H_9 + H_{10} + H_{11} + H_{12} + H_{13} + H_{14} + H_{15}$

$H_{total} = 0 + 0 + 13(4)$

$\boxed{H_{total} = 52 \text{ bits/frame}}$

Average rate of informatio

$R_s = r_s H(s)$

$r_s = 100 \text{ frames/sec}$

$R(s) = \text{frames/sec} * \text{bits/frame}$

$R(s) = 100 * 52$

$$\boxed{R(s) = 5200 \text{ bits/sec}}$$

(12.) A facimile transmission of a picture, there are ~~difficult~~ 4×10^6 pixels / frame. For a good reconstruction of the image, atleast 8 brightness levels are necessary. Assuming all levels are equally likely. Find the average information rate if 1 frame is transmitted every 4 sec.

Ans:- $H(s)$

$H_{total} =$

2^3

$2 \square \rightarrow 3$

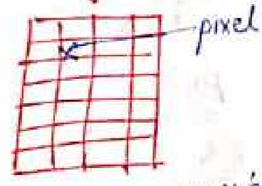
$\square^3 \rightarrow$ 0 or 1

The number of frames equals $= (8)^{4 \times 10^6} = N$

Similarly

$H(s)_{max} = \log_2 N$

$= \log_2 (4096 * 10^6)$



$= \underbrace{4 \times 10^6 \log_2 8}$

$= 12 \times 10^6 \text{ bits/}\cancel{\text{pixel}}\text{ frame}$

frame $= (8)^{4 \times 10^6}$

8

$r_s = 1/4$ frame/sec

$8^{4 \times 10^6}$

$R_s = r_s H(s)$

$= \frac{1}{4} * 12 \times 10^6$

2^{8} ^{32}

$$\boxed{R_s = 3 \text{ M bits/sec}}$$

(13.) A Black & white TV picture consists of 525 lines of picture information. Assume each line consists of 525 picture elements (pixel) and that each element can have 256 Brightness levels. Pictures are repeated at the rate of 30 frames/ sec. Calculate the average rate of information conveyed.

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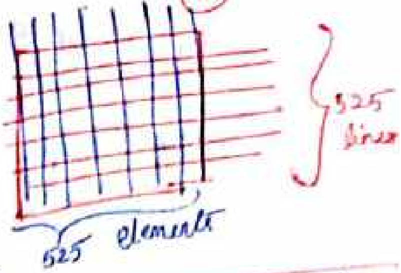
Ans: The number of frames $= 256^{525*525}$

$= 256^{525*525}$



$\therefore$ No. of pixels $= 525*525$

$$H(S)_{max} = \log_2 N$$

$$= \log_2 (256)^{525*525}$$

$$= 525*525 \log_2 (256)$$

$$\boxed{H(S)_{max} = 2205 \times 10^3 \text{ bits/frame}}$$

$$r_s = 30 \text{ frames/sec}$$

$$R_s = r_s * H(S)_{max}$$

$$\boxed{R_s = 66.15 \text{ M bits/sec}}$$

14. Shortly before a Horse race, A BookMaker believes that several horses have entered the race & have the following probability of winning

Horse	A	B	C	D	E
P(winning)	0.04	0.42	0.31	0.12	0.11

He then receives a message that due to injury one of the horse is not participating in the race. Explain how to access from an information theory point of view, the information value of this message

(i) If the horse is known.

(ii) If the horse is unknown.

Ans: Case (i) If the horse is known.

Since the horse that is injured is known, it could be A, B, C, D or E. Therefore the corresponding information would be

I_A, I_B, I_C, I_D (&) I_E respectively.

$$I_i = \log_2\left(\frac{1}{p_i}\right) \text{ bits}$$

$$I_A = \log_2\left(\frac{1}{0.04}\right) = 4.6439 \text{ bits} \quad \boxed{I_A = 4.6439 \text{ bits}}$$

$$I_B = \log_2\left(\frac{1}{0.42}\right) = 1.2515 \text{ bits} \quad \boxed{I_B = 1.2515 \text{ bits}}$$

$$I_C = \log_2\left(\frac{1}{0.31}\right) = 1.6897 \text{ bits} \quad \boxed{I_C = 1.6897 \text{ bits}}$$

$$I_D = \log_2\left(\frac{1}{0.12}\right) = 3.0589 \text{ bits} \quad \boxed{I_D = 3.0589 \text{ bits}}$$

$$I_E = \log_2\left(\frac{1}{0.11}\right) = 3.1844 \text{ bits} \quad \boxed{I_E = 3.1844 \text{ bits}}$$

Case (ii) Since the Horse that got injured is unknown. The only info. we get is the entropy.

$$H(S) = \sum_{i=1}^{5} p_i \log_2\left(\frac{1}{p_i}\right)$$

$$H(S) = 0.04\log_2\left(\frac{1}{0.04}\right) + 0.42\log_2\left(\frac{1}{0.42}\right) + 0.31\log_2\left(\frac{1}{0.31}\right) + 0.12\log_2\left(\frac{1}{0.12}\right) + 0.11\log_2\left(\frac{1}{0.11}\right)$$

$$\boxed{H(S) = 1.9525 \text{ BP Horse}}$$

EXTENSION OF ZERO MEMORY SOURCE

Consider a zero Memory Source emitting 2 symbols S_1 and S_2 with probabilities P_1 and P_2 respectively. Therefore the entropy will be $H(S)$

$$H(S) = P_1 \log_2\left(\frac{1}{P_1}\right) + P_2 \log_2\left(\frac{1}{P_2}\right) \quad -(1)$$ 12

Consider the 2nd extension of the source S denoted by S^2, then we have 4 possible combinations

s_1s_1, s_1s_2, s_2s_1, s_2s_2.

with probabilities

P_1^2, P_1P_2, P_2P_1 & P_2^2 respectively.

Also $\quad P_1^2 + P_1P_2 + P_2P_1 + P_2^2 = 1$

The entropy of the 2nd extension will be

$$H(S^2) = P_1^2 \log_2\left(\frac{1}{P_1^2}\right) + P_1P_2 \log_2\left(\frac{1}{P_1P_2}\right) + P_2P_1 \log_2\left(\frac{1}{P_2P_1}\right) + P_2^2 \log_2\left(\frac{1}{P_2^2}\right)$$

$$H(S^2) = 2P_1^2 \log_2\left(\frac{1}{P_1}\right) + 2P_1P_2 \log_2\left(\frac{1}{P_1P_2}\right) + 2P_2^2 \log_2\left(\frac{1}{P_2}\right)$$

$$H(S^2) = 2P_1^2 \log_2\left(\frac{1}{P_1}\right) + 2P_1P_2 \log_2\left(\frac{1}{P_1}\right) + 2P_1P_2 \log_2\left(\frac{1}{P_2}\right) + 2P_2^2 \log_2\left(\frac{1}{P_2}\right)$$

$$H(S^2) = 2P_1 \log_2\left(\frac{1}{P_1}\right)\left[P_1 + P_2\right] + 2P_2 \log_2\left(\frac{1}{P_2}\right)\left[P_1 + P_2\right]$$

$$H(S^2) = (P_1 + P_2)\left[2P_1 \log_2\left(\frac{1}{P_1}\right) + 2P_2 \log_2\left(\frac{1}{P_2}\right)\right]$$

$$H(S^2) = 2(P_1 + P_2)\left[P_1 \log_2\left(\frac{1}{P_1}\right) + P_2 \log_2\left(\frac{1}{P_2}\right)\right]$$

WKT $\boxed{P_1 + P_2 = 1}$

$$H(S^2) = 2\left[P_1 \log_2\left(\frac{1}{P_1}\right) + P_2 \log_2\left(\frac{1}{P_2}\right)\right]$$

$$\boxed{H(S^2) = 2\,H(S)}$$

15. Consider a Zero Memory source emitting 3 symbols X, Y & Z with respective probabilities 0.6, 0.3, & 0.1.

Calculate (i) Entropy of the source

(ii) All the symbols & their corresponding probabilities for second extension, also find the entropy.

(iii) Prove that $H(S^2) = 2\,H(S)$

Ans (i) $H(S) = \sum_{i=1}^{N} P_i \log_2\left(\frac{1}{P_i}\right)$

$$H(S) = 0.6 \log_2\left(\frac{1}{0.6}\right) + 0.3 \log_2\left(\frac{1}{0.3}\right) + 0.1 \log_2\left(\frac{1}{0.1}\right)$$

$$\boxed{H(S) = 1.2955 \text{ Bits / symbols}}$$

(ii)

X
$P(XX) = 0.6 * 0.6 = 0.36$
$P(XY) = 0.6 * 0.3 = 0.18$
$P(XZ) = 0.6 * 0.1 = 0.06$

Y
$P(YX) = 0.3 * 0.6 = 0.18$
$P(YY) = 0.3 * 0.3 = 0.09$
$P(YZ) = 0.3 * 0.1 = 0.03$

Z
$P(ZX) = 0.1 * 0.6 = 0.06$
$P(ZY) = 0.1 * 0.3 = 0.03$
$P(ZZ) = 0.1 * 0.1 = 0.01$

$$H(s^2) = P(XX)\log_2\left(\frac{1}{P(XX)}\right) + P(XY)\log_2\left(\frac{1}{P(XY)}\right) + P(XZ)\log_2\left(\frac{1}{P(XZ)}\right) + P(YX)\log_2\left(\frac{1}{P(YX)}\right) + P(YY)\log_2\left(\frac{1}{P(YY)}\right) + P(YZ)\log_2\left(\frac{1}{P(YZ)}\right) + P(ZX)\log_2\left(\frac{1}{P(ZX)}\right) + P(ZY)\log_2\left(\frac{1}{P(ZY)}\right) + P(ZZ)\log_2\left(\frac{1}{P(ZZ)}\right)$$

$$H(s^2) = 0.36\log_2\left(\frac{1}{0.36}\right) + 0.18\log_2\left(\frac{1}{0.18}\right) + 0.06\log_2\left(\frac{1}{0.06}\right) + 0.18\log_2\left(\frac{1}{0.18}\right) + 0.09\log_2\left(\frac{1}{0.09}\right) + 0.03\log_2\left(\frac{1}{0.03}\right) + 0.06\log_2\left(\frac{1}{0.06}\right) + 0.03\log_2\left(\frac{1}{0.03}\right) + 0.01\log_2\left(\frac{1}{0.01}\right)$$

$$H(s^2) = +0.5306 + 0.4453 + 0.2435 + 0.4453 + 0.3127 + 0.1518 + 0.2435 + 0.1518 + 0.0664$$

$$\boxed{H(s^2) = 2.5909 \text{ Bits/Symbol}}$$

(iii) TO PROVE.

$H(s^2) = 2H(s)$

$H(s) = 1.2955$ bits/symbol

$2H(s) = 2.5909$ bits/symbol

$\therefore \boxed{H(s^2) = 2H(s)}$ H.P

CARDS

52 → Spade, Club, Diamond, Heart

13 cards each

A 2 3 4 5 6 7 8 9 10 J Q K

(17) An analog signal is bandlimited to 500Hz & is sampled (14) at the Nyquist Rate. The samples are quantized into (4) levels. The Quantization levels are assumed to be independent. & occur with probability $P_1 = P_4 = 1/8$, $P_2 = P_3 = 3/8$. Find the information rate.

Ans:- $H(S) = \sum_{i=1}^{4} P_i \log_2\left(\frac{1}{P_i}\right)$

$$H(S) = \frac{1}{8}\log_2\left(\frac{1}{1/8}\right) + \frac{3}{8}\log_2\left(\frac{1}{3/8}\right) + \frac{1}{8}\log_2\left(\frac{1}{1/8}\right) + \frac{3}{8}\log_2\left(\frac{1}{3/8}\right)$$

$$\boxed{H(S) = 1.8113 \text{ bits/~~samples~~ level.}}$$

$R(S) = r(S)\, H(S)$

$r(S) = 2 * 500\text{Hz} = 1000\text{Hz}$ [$\because$ Nyquist Rate]

$R(S) = 1000 * 1.8113$

$$\boxed{R(S) = 1811.27 \text{ bits/sec}}$$

30/08/18

n^{th} EXTENSION SOURCE

TO PROVE THAT

$H(S^n) = n\,H(S)$

PROOF: Consider a source with symbols $S = \{S_1, S_2, \ldots S_N\}$ with probabilities $P = \{P_1, P_2, \ldots \ldots P_N\}$. The n^{th} extension S^n will have N^n symbols.

σ_{N^n}.

$$S^n = \{\sigma_1, \sigma_2, \ldots \ldots \sigma_{N^n}\}$$

$$P(\sigma_1) = \{P(S_{i_1}), P(S_{i_2}), \ldots \ldots P(S_{i_n})\}$$

$$\sum_{S^n} P(\sigma_i) = \sum_{i=1}^{N^n} \Big| P(S_i) P(S_{i_2}) P(S_{i_3}) \cdots P(S_{i_n}) \Big|$$

$$\sum_{S^n} P(\sigma_i) = \sum_{i=1}^{N} P(S_i) \sum_{i=1}^{N} P(S_{i_2}) \sum_{i=1}^{N} P(S_{i_3}) \cdots \sum_{i=1}^{N} P(S_{i_n})$$

$$= 1 * 1 * 1 * 1 * 1$$

$$\sum_{S^n} P(\sigma_i) = 1 \quad \text{—— (1)}$$

The entropy

$$H(S^n) = \sum_{S^n} P(\sigma_i) \log_2 \left[\frac{1}{P(\sigma_i)}\right]$$

$$H(S^n) = \sum_{S^n} P(\sigma_i) \log_2 \left[\frac{1}{P(S_{i_1}) P(S_{i_2}) P(S_{i_3}) \cdots P(S_{i_n})}\right]$$

$$H(S^n) = \sum_{S^n} P(\sigma_i) \log_2 \left(\frac{1}{P(S_{i_1})}\right) + \sum_{S^n} P(\sigma_i) \log_2 \left(\frac{1}{P(S_{i_2})}\right) \cdots \cdots \quad \text{—— (2)}$$

Consider

$$\underbrace{\sum_{S^n} P(\sigma_i) \log_2 \left[\frac{1}{P(S_{i_1})}\right]} = \sum_{i=1}^{N} P(S_{i_1}) \log_2 \frac{1}{P(S_{i_1})}$$

from (1)

$$* \sum_{i=1}^{N} P(S_{i_2}) \cancel{\log_2 \frac{1}{P(S_{i_2})}} *$$

$$\cdots \sum_{i=1}^{N} P(S_{i_3}) \cdots$$

$H(s^2) = 2H(s)$

Important Notes

$= H(s) + 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1$

$= H(s)$

Similarly

consider the above result in eqn ②

$H(S^n) = H(s) + H(s) + H(s) \cdot \cdot \cdot \cdot \cdot + H(s)$

$$\boxed{H(S^n) = n\, H(s)}$$

Important Notes

AVERAGE INFORMATION CONTENT IN LONG DEPENDENT SEQUENCES:-

A dependent probabilistic model is required to model sources in which symbol emission depends on previous symbol.

A model which is used to represent source with memory is called Markoff (or) Markov. Model.

In general N^{th} order Markoff's Source, the emission of symbol S at the current instant is dependent on its previous ~~N~~ n Symbols.

MARKOFF MODEL :

A system with memory can be represented using a state diagram. A state diagram represents all possible states along with the transition probabilities.

Also the symbols emitted by the source in each transition are shown in the state diagram.

The Tree diagram can be constructed from the state diagram, the probabilities of symbol emitted by the source can be determined from the Tree diagram.

The probabilities of message of length 'l' can be determined by constructing a Tree-Diagram of 'l' stages.

ENTROPY AND INFORMATION RATE :-

Let entropy of the state 'k' be denoted by H_k. It can be obtained by considering all the outgoing probabilities of state 'k'.

*** $$H_k = \sum_{l=1}^{M} P_{kl} \log_2 \left(\frac{1}{P_{kl}}\right) \text{ bits/symbol}$$

$M \rightarrow$ No. of states

The entropy of the source is

$$\boxed{H = \sum_{k=1}^{M} H_k P_k \quad \text{bits/symbol}}$$

The average information rate

$$\boxed{R(s) = H \cdot r_s.}$$

AVERAGE INFORMATION PER SYMBOL :-

The average information content per symbol in a message of length 'N' is given by

$$\boxed{G_N = \frac{1}{N} \sum_{\forall i} P(m_i) \log_2\left(\frac{1}{P(m_i)}\right) \text{ bits/symbol}}$$

$P(m_i) \rightarrow$ probability of message whose length is 'l'.

$$G_N \rightarrow \frac{1}{N} H(\bar{S}^N)$$

$H(\bar{S}) \rightarrow$ Entropy of adjoint source.

The average amount of information for a system in long messages decreases with increase in 'N' and will be atleast equal to $H(s)$.

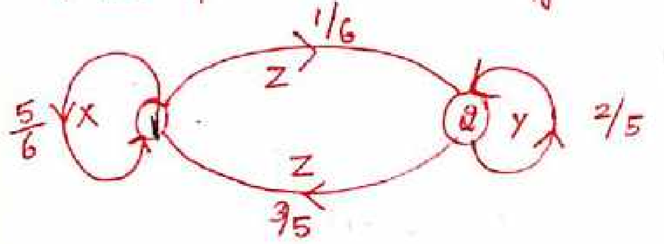
$$G_1 \geq G_2 \geq G_3 > \cdots \geq H(s)$$

$$\lim_{N\to\infty} G_N = H(s)$$

18. For the given state diagram, find

a) Entropy of each state & Total entropy.

b) Find G_1 and G_2. Also verify $G_1 > G_2 > H$.



Ans:- (a)

$$H_k = \sum_{i=1}^{M} P_{kl} \log_2\left(\frac{1}{P_{kl}}\right)$$

From the state diagram, we have

$$P(1) = \frac{5}{6} P(1) + \frac{3}{5} P(2)$$

$$P(2) = \frac{2}{5} P(2) + \frac{1}{6} P(1)$$

$$\frac{1}{6} P(1) - \frac{3}{5} P(2) = 0$$

$$-\frac{1}{6} P(1) + \frac{3}{5} P(2) = 0$$

$$\cancel{P(1) = 0 \quad P(2) = 0}$$

$$\frac{1}{6} P(1) = \frac{3}{5} (P(2)$$

$$P(1) + P(2) = 1$$

$$\boxed{P(1) = \frac{18}{23}, \quad P(2) = \frac{5}{23}.}$$

a) To find H_1, H_2, & H.

$$H_1 = \sum_{i=1}^{2} P_{1l} \log_2\left(\frac{1}{P_{1l}}\right)$$

$$= P_{11} \log_2\left(\frac{1}{P_{11}}\right) + P_{12} \log_2\left(\frac{1}{P_{12}}\right)$$

$$= \frac{5}{6} \log_2\left(\frac{6}{5}\right) + \frac{1}{6} \log_2 (6)$$

$$\boxed{H_1 = 0.6500 \text{ BPS}}$$

(17)

$$H_2 = \sum_{i=1}^{2} P_{2i} \log_2\left(\frac{1}{P_{2i}}\right)$$

$$= P_{21} \log_2\left(\frac{1}{P_{21}}\right) + P_{22} \log_2\left(\frac{1}{P_{22}}\right)$$

$$= \frac{3}{5} \log_2\left(\frac{5}{3}\right) + \frac{2}{5} \log_2\left(\frac{5}{2}\right)$$

$$\boxed{H_2 = 0.9710 \text{ BPS}}$$

$$H = \sum_{k=1}^{M} H_k P_k$$

$$H = \sum_{k=1}^{2} H_k P_k$$

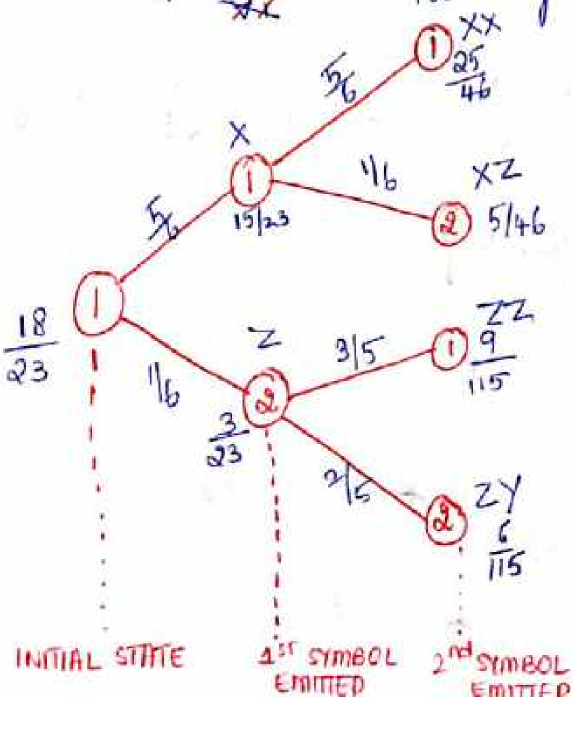
$$H = H_1 P(1) + H_2 P(2)$$

$$H = 0.65\left(\frac{18}{23}\right) + 0.9710\left(\frac{5}{23}\right)$$

$$\boxed{H = 0.7198 \text{ BPS}}$$

(b) $$G_N = \frac{1}{N} \sum_{\forall i} P(m_i) \log_2\left(\frac{1}{P(m_i)}\right)$$

To find G_1 and G_2, we need to construct the Tree diagram.



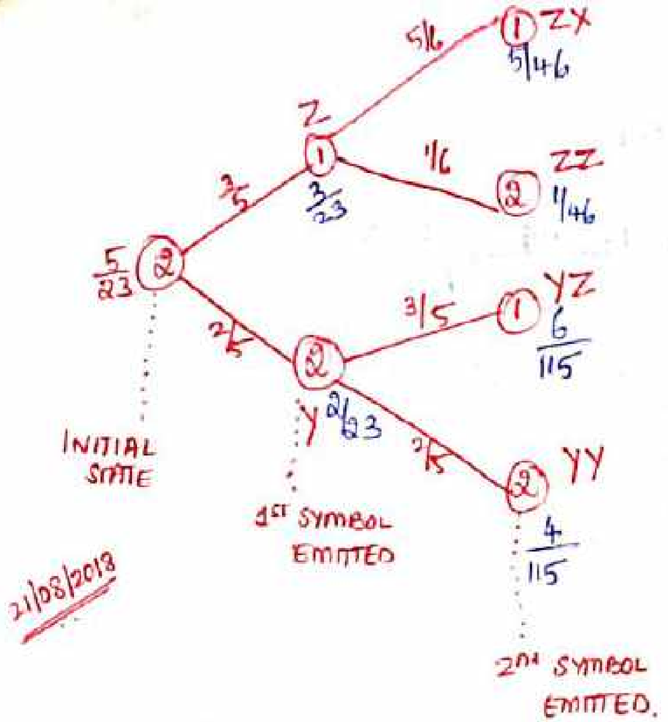
$$P(X) = \frac{18}{23} \times \frac{5}{6} = \frac{15}{23}.$$

$$P(Z) = \left.\begin{matrix}\frac{18}{23} \times \frac{1}{6} \\ + \frac{5}{23} \times \frac{3}{5}\end{matrix}\right\} = \frac{3}{23} + \frac{3}{23} = \frac{6}{23}$$

$$P(XX) = \frac{18}{23} * \frac{5}{6} * \frac{5}{6} = \frac{25}{46}$$

$$P(XZ) = \frac{18}{23} * \frac{5}{6} * \frac{1}{6} = \frac{5}{46}$$

$$P(ZZ) = \frac{18}{23} * \frac{1}{6} * \frac{3}{5} + \frac{5}{23} * \frac{3}{5} * \frac{1}{6} = \frac{1}{10}$$



~~$P(X) = \frac{15}{23}$~~

$$P(ZY) = \frac{18}{23} * \frac{1}{6} * \frac{2}{5} = \frac{6}{115}$$

$$P(ZX) = \frac{5}{23} * \frac{3}{5} * \frac{5}{6} = \frac{5}{46}$$

$$P(YZ) = \frac{5}{23} * \frac{2}{5} * \frac{3}{5} = \frac{6}{115}$$

$$P(YY) = \frac{5}{23} * \frac{2}{5} * \frac{2}{5} = \frac{4}{115}$$

$$P(Y) = \frac{5}{23} * \frac{2}{5} = \frac{2}{23}$$

$P(X) = 15/23$	,	$P(ZY) = 6/115$
$P(Z) = 6/23$	,	$P(Y) = 2/23$
$P(XX) = 25/46$	,	$P(ZX) = 5/46$
$P(XZ) = 5/46$	,	$P(YZ) = 6/115$
$P(ZZ) = \frac{1}{10}$	,	$P(YY) = \frac{4}{115}$

b) $$G_N = \frac{1}{N} \sum_{\forall i} P(m_i) \log_2 \left(\frac{1}{P(m_i)} \right)$$

$$G_1 = \frac{1}{1} \left[P(x) \log_2 \left(\frac{1}{P(x)} \right) + P(Y) \log_2 \left(\frac{1}{P(Y)} \right) + P(Z) \log_2 \left(\frac{1}{P(Z)} \right) \right.$$

$$G_1 = \frac{15}{23} \log_2 \left(\frac{23}{15} \right) + \frac{2}{23} \log_2 \left(\frac{23}{2} \right) + \frac{6}{23} \log_2 \left(\frac{23}{6} \right)$$

$$\boxed{G_1 = 1.2143 \text{ Bits/symbol}}$$

$$G_2 = \frac{1}{2} \left[P(XX) \log_2 \left(\frac{1}{P(XX)} \right) + P(XZ) \log_2 \left(\frac{1}{P(XZ)} \right) + P(ZZ) \log_2 \left(\frac{1}{P} \right. \right.$$

$$+ P(ZY) \log_2 \left(\frac{1}{P(ZY)} \right) + P(ZX) \log_2 \left(\frac{1}{P(ZX)} \right) +$$

$$P(YZ) \log_2 \frac{1}{P(YZ)} + P(YY) \log_2 \left(\frac{1}{P(YY)} \right)$$

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$$G_2 = \frac{1}{2}\left\{\frac{25}{46}\log_2\left(\frac{46}{25}\right) + \frac{5}{46}\log_2\left(\frac{46}{5}\right) + \frac{1}{10}\log_2(10) + 18\right.$$

$$\frac{6}{115}\log_2\left(\frac{115}{6}\right) + \frac{5}{46}\log_2\left(\frac{46}{5}\right) + \frac{6}{115}\log_2\left(\frac{115}{6}\right) +$$

$$\left. + \frac{4}{115}\log_2\frac{115}{4}\right\}$$

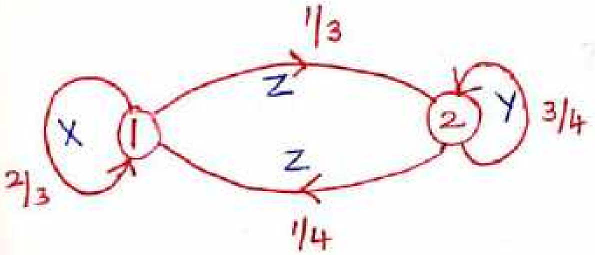
$$\boxed{G_2 = 1.0597 \text{ bits/symbol}}$$

Hence its Verified :-

$G_1 > G_2 > H$ as $1.2143 > 1.0597 > 0.7198$

Hence proved

(21) For the given state diagram, calculate total entropy & prove that $G_1 > G_2 > G_3 > H$



Ans From this state diagram

$P(1) + P(2) = 1$

$$P(1) = \frac{2}{3}P(1) + \frac{1}{4}P(2)$$

$$P(2) = \frac{3}{4}P(2) + \frac{1}{3}P(1)$$

$$\frac{1}{3}P(1) = \frac{1}{4}P(2)$$

$$P(1) = \frac{3}{7} \qquad P(2) = \frac{4}{7}$$

(22) a) To Find H_1, H_2 & H

$$H_k = \sum_{\ell=1}^{M} P_{k\ell}\log_2\left(\frac{1}{P_{k\ell}}\right)$$

$$H_1 = \sum_{\ell=1}^{2} P_{1\ell}\log_2\left(\frac{1}{P_{1\ell}}\right)$$

$$H_1 = P_{11}\log_2\left(\frac{1}{P_{11}}\right) + P_{12}\log_2\left(\frac{1}{P_{12}}\right)$$

$$H_1 = \frac{2}{3}\log_2\left(\frac{3}{2}\right) + \frac{1}{3}\log_2(3)$$

$$\boxed{H_1 = 0.9183 \text{ BPS}}$$

$$H_2 = \sum_{l=1}^{2} P_{2l}\log_2\left(\frac{1}{P_{2l}}\right)$$

$$H_2 = P_{21}\log_2\left(\frac{1}{P_{21}}\right) + P_{22}\log_2\left(\frac{1}{P_{22}}\right)$$

$$H_2 = \frac{1}{4}\log_2(4) + \frac{3}{4}\log_2\left(\frac{4}{3}\right)$$

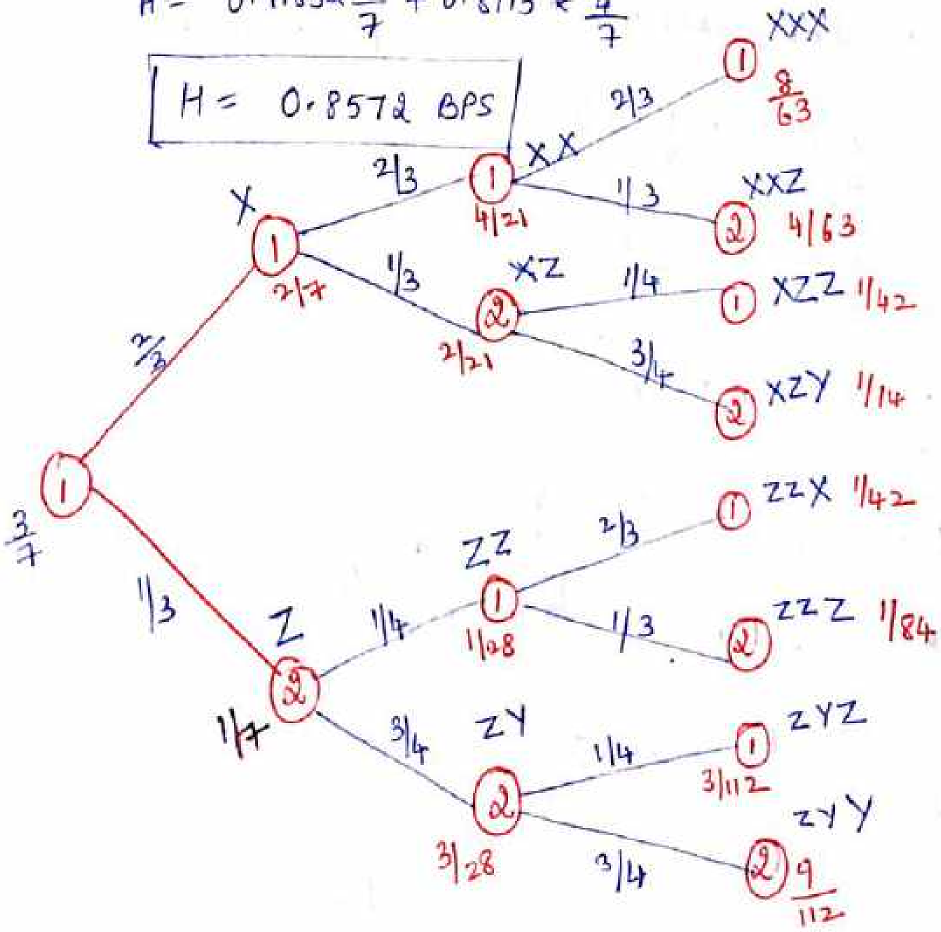
$$\boxed{H_2 = 0.8113 \text{ BPS}}$$

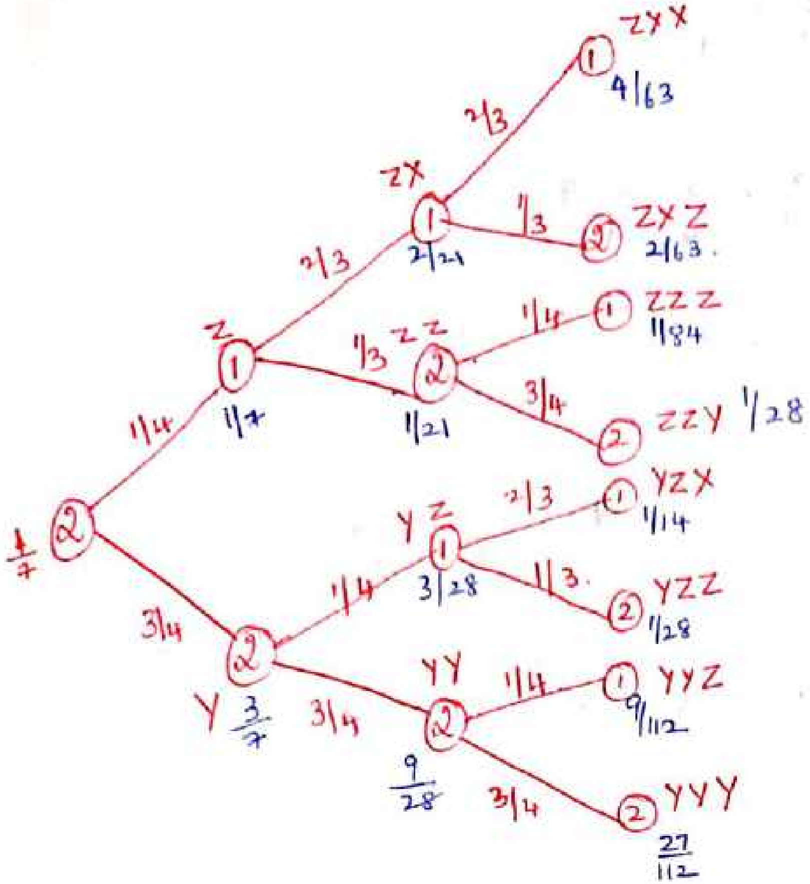
$$H = \sum_{R=1}^{M} H_R P_R \quad \underline{\text{BPS}}$$

$$H = H_1 P(1) + H_2 P(2)$$

$$H = 0.9183 \times \frac{3}{7} + 0.8113 \times \frac{4}{7}$$

$$\boxed{H = 0.8572 \text{ BPS}}$$





$P(X) = \frac{3}{7} * \frac{2}{3} = \frac{2}{7}$

$P(Y) = \frac{4}{7} \times \frac{3}{4} = \frac{3}{7}$

$P(Z) = \frac{3}{7} \times \frac{1}{3} + \frac{4}{7} \times \frac{1}{4} = \frac{2}{7}$

$P(XX) = \frac{3}{7} \times \frac{2}{3} \times \frac{2}{3} = \frac{4}{21}$

$P(XZ) = \frac{3}{7} \times \frac{2}{3} \times \frac{1}{3} = \frac{2}{21}$

$P(ZZ) = \frac{3}{7} \times \frac{1}{3} \times \frac{1}{4} + \frac{4}{7} \times \frac{1}{4} \times \frac{1}{3} = \cancel{\frac{7}{84}} \ \frac{1}{12}$

$P(ZY) = \frac{3}{7} \times \frac{1}{3} \times \frac{3}{4} = \frac{3}{28}$

$P(ZX) = \frac{4}{7} \times \frac{1}{4} \times \frac{2}{3} = \frac{2}{21}$

$P(YZ) = \frac{4}{7} \times \frac{3}{4} \times \frac{1}{4} = \frac{3}{28}$

$P(YY) = \frac{4}{7} \times \frac{3}{4} \times \frac{3}{4} = \frac{9}{28}$

$P(XXX) = \frac{3}{7} \times \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{8}{63}$

$P(XXZ) = \frac{3}{7} \times \frac{2}{3} \times \frac{2}{3} \times \frac{1}{3} = \frac{4}{63}$

$P(XZZ) = \frac{3}{7} \times \frac{2}{3} \times \frac{1}{3} \times \frac{1}{4} = \frac{1}{42}$

$P(XZY) = \frac{1}{14}$

$P(ZZX) = \frac{1}{42}$

$P(ZZZ) = \frac{1}{84} + \frac{1}{84} = \frac{2}{84} = \frac{1}{42}$

$P(ZYZ) = \frac{3}{112} +$

$P(ZYY) = \frac{9}{112} +$

$P(ZXX) = \frac{4}{63}$

$P(ZXZ) = 2/63$

$P(ZZY) = 1/28$

$P(YZX) = 1/14$

$P(YZZ) = 1/28$

$P(YYZ) = 9/112$

$P(YYY) = 27/112$

b) $$G_N = \frac{1}{N} \sum_{\forall i} P(m_i) \log_2\left(\frac{1}{P(m_i)}\right)$$

$$G_1 = \frac{1}{1} \; P(X) \log_2\left(\frac{1}{P(X)}\right) + P(Y) \log_2\left(\frac{1}{P(Y)}\right) + P(Z) \log_2\left(\frac{1}{P(Z)}\right)$$

$$= \frac{2}{7} \log_2\left(\frac{7}{2}\right) + \frac{3}{7} \log_2\left(\frac{7}{3}\right) + \frac{2}{7} \log_2\left(\frac{7}{2}\right)$$

$$\boxed{G_1 = 1.5567 \text{ bits/symbol}}$$

$$G_2 = \frac{1}{2}\Big\{ P(XX) \log_2\left(\frac{1}{P(XX)}\right) + P(XZ) \log_2 \frac{1}{P(XZ)} + P(ZZ) \log_2 \frac{1}{P(ZZ)}$$

$$+ P(ZY) \log_2 \frac{1}{P(ZY)} + P(ZX) \log_2 \frac{1}{P(ZX)}$$

$$+ P(YZ) \log_2 \frac{1}{P(YZ)} + P(YY) \log_2 \frac{1}{P(YY)}\Big\}$$

$$G_2 = \frac{1}{2}\Big\{ \frac{4}{21} \log_2\left(\frac{21}{4}\right) + \frac{2}{21} \log_2\left(\frac{21}{2}\right) + \frac{1}{12} \log_2(12)$$

$$+ \frac{3}{28} \log_2\left(\frac{28}{3}\right) + \frac{2}{21} \log_2\left(\frac{21}{2}\right) +$$

$$\frac{3}{28} \log_2\left(\frac{28}{3}\right) + \frac{9}{28} \log_2\left(\frac{28}{9}\right).$$

$$G_2 = 1.3087 \text{ bits/symbol}$$

$$G_3 = \frac{1}{3}\Big\{ P(XXX) \log_2 \frac{1}{P(XXX)} + P(XXZ) \log_2 \frac{1}{P(XXZ)} + P(XZZ) \log_2 \frac{1}{P(XZZ)}$$

$$+ P(XZY) \log_2 \frac{1}{P(XZY)} + P(ZZX) \log_2 \frac{1}{P(ZZX)} +$$

$$+ P(ZZZ)\log_2 \frac{1}{P(ZZZ)} + P(ZYZ)\log_2 \frac{1}{P(ZYZ)} + P(ZYY)\log_2 \frac{1}{P(ZYY)}$$

$$+ P(ZXX)\log_2 \frac{1}{P(ZXY)} + P(ZXZ)\log_2 \frac{1}{P(ZXZ)} + P(ZXX)\log_2 \frac{1}{P(ZXX)}$$

$$+ P(ZXZ)\log_2 \frac{1}{P(ZXZ)} + P(ZZY)\log_2 \frac{1}{P(ZZY)} + P(YZX)\log_2 \frac{1}{P(YZX)}$$

$$+ P(YZZ)\log_2 \frac{1}{P(YZZ)} + P(YYZ)\log_2 \frac{1}{P(YYZ)} + P(YYY)\log_2 \frac{1}{P(YYY)}$$

$$G_3 = \frac{1}{3}\Big\{ \frac{8}{63}\log_2\left(\frac{63}{8}\right) + \frac{4}{63}\log_2\left(\frac{63}{4}\right) + \frac{1}{42}\log_2(42) +$$
$$\frac{1}{14}\log_2(14) + \frac{1}{42}\log_2(42) + \frac{1}{42}\log_2(42) +$$
$$\frac{3}{112}\log_2\left(\frac{112}{3}\right) + \frac{9}{112}\log_2\left(\frac{112}{9}\right) + \frac{4}{63}\log_2\left(\frac{63}{4}\right) +$$
$$\frac{2}{63}\log_2\left(\frac{63}{2}\right) + \frac{1}{28}\log_2(28) + \frac{1}{14}\log_2(14) +$$
$$\frac{1}{28}\log_2(28) + \frac{9}{112}\log_2\left(\frac{112}{9}\right) + \frac{27}{112}\log_2\left(\frac{112}{27}\right)\Big\}$$

$$\boxed{G_3 = \cancel{1.2201}\ 1.1776 \text{ bits/symbol}}$$

$G_1 > G_2 > G_3 > H$ as

$1.5567 > 1.3087 > \cancel{1.2201}\ 1.1776 > 0.8572$

HP

21/08/18

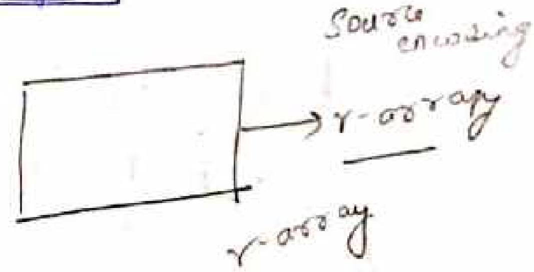
MODULE-2

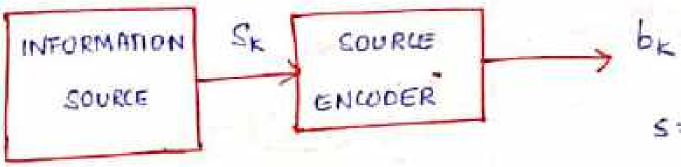
SOURCE ENCODING

Source Encoding is a process by which output of an information source is converted into r-array sequence, where

r → is the number of different symbols used in the transformation source.

Eg:- ADC





$S = \{S_1, S_2, \ldots \ldots S_N\}$

$X = \{x_1, x_2, \ldots \ldots x_K\}$

$r=2 \rightarrow 0, 1$ Binary

$r=3 \rightarrow 0, 1, 2$ Ternary

$r=4 \rightarrow 0, 1, 2, 3$ Quaternary

Source encoder is used to ensure better compression of input symbols. Compression could be lossless or lossy depending on the applications.

TYPES OF CODES :-

(i) BLOCK CODES :- Each symbol will be mapped onto a block of count symbols defined in the code alphabet. Block codes can be either fixed or variable length.

eg. $S = \{S_1, S_2, S_3\}$

	Code A	Code B
S_1	0 0	0
S_2	0 1	0 1
S_3	1 1	0 1 1

(2) NON-SINGULAR CODES :-

All the code words are distinct and easily distinguishable.

Eg:- (1)

	Non-singular Code A	Singular Code B
S_1	0 0	1 1
S_2	0 1	1 0
S_3	1 0	1 1
S_4	1 1	0 0 1

Same code word for 2 diff symbols.

(2)

	1st Extension
S_1	0
S_2	0 0
S_3	0 1
S_4	1 1

2nd extension

$S_1 S_1 = 00$
$S_1 S_2 = 000$
$S_1 S_3 = 001$
$S_1 S_4 = 011$
$S_2 S_1 = 000$
$S_2 S_2 = 0000$
$S_2 S_3 = 0001$
$S_2 S_4 = 0011$
$S_3 S_1 = 010$
$S_3 S_2 = 0100$
$S_3 S_3 = 0101$
$S_3 S_4 = 0111$
$S_4 S_1 = 110$
$S_4 S_2 = 1100$
$S_4 C_3 = 1101$
$C_4 S_4 = 1111$

singular

The 1st extension is Non-singular whereas 2nd extension is Singular.

(3). UNIQUELY DECODABLE CODES :-

A Block code is said to be uniquely decodable if its N^{th} extension is also non-singular. for finite values of n.

Eg:-

	Code A	Code B
S_1	0 0	0
S_2	0 1	1 0
S_3	1 0	1 1 0
S_4	1 1	1 1 1

(4) INSTANTANEOUS CODE

A uniquely decodable code is said to be instantaneous if it is possible to identify the end of code-word in any received sequence without reference with to be succeeding symbols.

i.e; there is No time delay in the process of decoding.

eg.

	Code A	Code B
S_1	0	$\boxed{0}$
S_2	0 1	1 $\boxed{0}$
S_3	0 1 1	1 1 $\boxed{0}$
S_4	0 1 1 1	1 1 1 $\boxed{0}$

Consider the received sequence to be 001100.

Code B: 0 | 0 | 1 1 0 | 0 → S_1 | S_1 | S_3 | S_1

Code A: 0 | 0 1 1 | 0 | 0 → S_1 | S_3 | S_1 | S_1

Code B is instantaneous as a zero [0] indicates end of code word

(5) OPTIMAL CODE

Instantaneous code is said to be optimal if it has minimum length.

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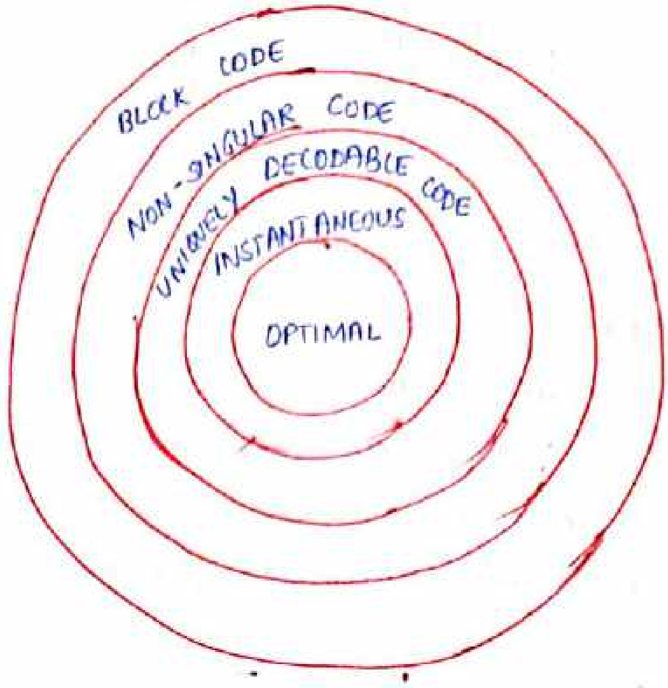

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6. Prefix Codes (Instantaneous codes)

Instantaneous code is one in which we can properly decode the information without any ambiguity (or) the need for additional information. This is the necessary and sufficient for the code to be instantaneous.

Ex:

SYMBOL	CODE WORD
A	0
B	01
C	11
D	10

Transmitted information – ABCAD

corresponding sequence – 0|01|11|0|10

Here decoding – AACDD

Therefore, decoding will not be successful if any of the code words start with any of the valid code words. Therefore the necessary condition of an instantaneous code is that, no code word should be prefix of any other code.

All prefix codes satisfy the property,

$$\sum_{i=1}^{N} \gamma^{-l_i} \le 1 \quad \text{KMI}$$

$$\sum_{i=1}^{N} \gamma^{-l_i} \le 1$$

where $\gamma = 2, 3, 4 \ldots$

l_i = length of code word i

N = No. of codewords in total.

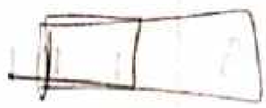
The above property is called as Kraft McMillan Inequality (KMI)

All prefix codes satisfy the above property, but all codes that satisfy the above property need not be prefix.

Problems

① Determine which code is prefix and also draw the decision diagram for a prefix code.

Code A	code B	code C	code D
0	1	00	10
10	01	110	111
110	111	1110	110
1110	10	001	01
111	00	011	00



Soln:-

	Code A	Length
S_1	0	$1 - l_1$
S_2	10	$2 - l_2$
S_3	110	$3 - l_3$
S_4	1110	$4 - l_4$
S_5	111	$3 - l_5$

In code A, $S_5 = 111$ which is prefix of $S_4 = 1110$. Hence it is not instantaneous code/prefix code.

The Kraft McMillan Inequality is given by,

$$\sum_{i=1}^{N} r^{-l_i} \leq 1$$

Here, $r = 2$ (0 or 1)

$$= 2^{-1} + 2^{-2} + 2^{-3} + 2^{-4} + 2^{-3}$$

$$= 1.0625 \nleq 1$$

By inspection, we found that its not a prefix code which is also been verified by KMI, as it does not satisfy the property.

b)

	Code B	Length
S_1	1	$1 - l_1$
S_2	0 1	$2 - l_2$
S_3	1 1 1	$3 - l_3$
S_4	1 0	$2 - l_4$
S_5	0 0	$2 - l_5$

$S_1 = 1$ is the prefix of $S_4 - 10$, $S_3 - 111$. Hence by inspection, Its not a Prefix code.

From Kraft Mc Millan's Inequality.

$$\sum_{i=1}^{N} r^{-l_i} \leq 1$$

Here $r = 2$

$$= 2^{-1} + 2^{-2} + 2^{-3} + 2^{-2} + 2^{-2}$$

$$= 1.3750 \nleq 1$$

$\therefore$ It does not satisfy the KMI property

$\therefore$ Hence not a prefix code.

c)

	Code C	length
S_1	00	$2 - l_1$
S_2	110	$3 - l_2$
S_3	1110	$4 - l_3$
S_4	001	$3 - l_4$
S_5	011	$3 - l_5$

$S_1 = 00$ is a prefix of $S_4 = 001$ $\therefore$ It is not a Prefix code.

From Kraft Mc Millan's property

$$\sum_{i=1}^{N} r^{-l_i} \leq 1$$

$r = 2$

$$= 2^{-2} + 2^{-3} + 2^{-4} + 2^{-3} + 2^{-3}$$

$$= 0.6875 \leq 1 \checkmark \text{ (Satisfied)}$$

Though the given code satisfies KMI, it is not a Prefix code.

d) Code - D

Name	Code - D	length
S_1	10	$2 - l_1$
S_2	111	$3 - l_2$
S_3	110	$3 - l_3$
S_4	01	$2 - l_4$
S_5	00	$2 - l_5$

By Inspection, there is no prefix in the given code word.

∴ It is a prefix code. But which may be verified KMI property,

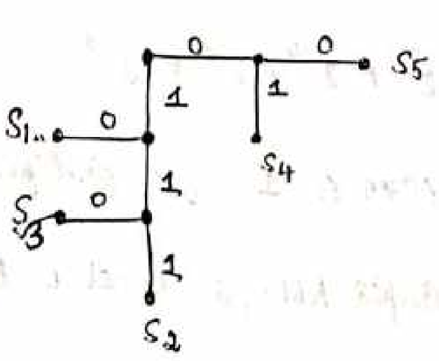
$$\sum_{i=1}^{N} r^{-l_i} \leq 1$$

$$= 2^{-2} + 2^{-3} + 2^{-3} + 2^{-2} + 2^{-2}$$

$$= 1 \leq 1$$

∴ Since it satisfies the KMI property. Its a Prefix code.

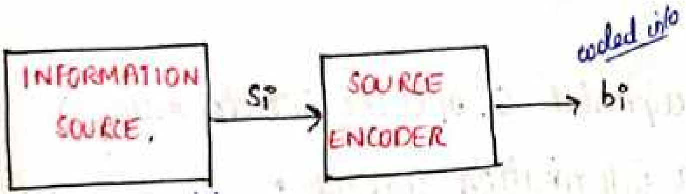
Code - Tree for Code - D



Write prefix code

10
11
20
1212
1201
2112
2220
2201

8. SOURCE CODING THEOREM



Consider an information source as shown in figure. Let s_i be the symbols emitted from the source. The output of the source is given to the source encoder whose output is the coded information b_i. Let l_i be the length for the i^{th} symbol s_i emitted by the source. The average code wordlength is given by

$$L = \sum_{i=1}^{N} P_i l_i$$

The efficiency of the source Encoder is given by

$$\eta_s = \frac{L_{min}}{L} \times 100$$

As the entropy gives the average information conveyed by the source, no symbol emitted from the source can be represented with lesser bits than that of its entropy,

i.e, $L \geq H(s)$

$L_{min} = H(s)$

Thus we require at least $H(s)$ numbers of bits to represent any of the symbols emitted by the source ... lossless communication, that is

$$\eta_s = \frac{H(s)}{L} \times 100$$

The redundancy of the source is

$$\boxed{R_s = 1 - \eta_s.}$$

SHANNON'S FIRST ENCODING THEOREM :

ENCODING PROCEDURE

STEP 1 : Given the source alphabet S and the corresponding probability P for a given information source.

STEP 2 : Arrange the probabilities in the non-increasing order (decreasing order).

STEP 3 : Compute the length l_i for the corresponding code word to each symbol s_i given by

★★★ $$\boxed{l_i \geq \log_2\left(\frac{1}{P_i}\right)} \text{ or } \boxed{2^{l_i} \geq \frac{1}{P_i}}$$

l_i is the smallest possible integer which satisfies the given condition.

STEP 4 : Define the following parameters from the probability set

$$q_1 = 0$$

$$q_2 = P_1 = q_1 + P_1$$

$$q_3 = P_1 + P_2 = q_2 + P_2$$

$$q_4 = P_1 + P_2 + P_3 = q_3 + P_3$$

$$\vdots$$

$q_{N+1} = 1$

where N is the total number of source symbols in the source alphabet.

STEP 5 - Expand q_i in binary till l_i number of places after the decimal point.

STEP 6 - The number after decimal places in binary representation of q_i are the code words for the corresponding symbol S_i.

29/08/2018

1. Consider a source with source alphabets $S = \underbrace{(A, B, C, D)}$. with corresponding probabilities $P = (0.1, 0.2, 0.3, 0.4)$

Find the code words for symbols using SHANNON'S ALGORITHM. Also find the source efficiency and redundancy.

ANS :- STEP 2

$P = (0.4, 0.3, 0.2, 0.1)$

$S = (D, C, B, A)$

STEP 3

$l_i \geq \log_2\left(\frac{1}{P_i}\right)$

$l_1 \geq \log_2\left(\frac{1}{P_1}\right) \geq 1.32$ $[P_1 = 0.4]$

$\therefore \boxed{l_1 = 2}$

$l_2 \geq \log_2\left(\frac{1}{P_2}\right) \geq 1.7370$ $[P_2 = 0.3]$

$\boxed{l_2 = 2}$

$l_3 \geq \log_2\left(\frac{1}{P_3}\right) \geq 2.3219$ $[P_3 = 0.2]$

$\boxed{l_3 = 3}$

$l_4 \geq \log_2\left(\frac{1}{P_4}\right) \geq 3.3219$ $[P_4 = 0.1]$

$\boxed{l_4 = 4}$

$\boxed{\begin{matrix} l_1 = 2 \\ l_2 = 2 \\ l_3 = 3 \\ l_4 = 4 \end{matrix}}$

Step (4)

$q_1 = 0$

$q_2 = q_1 + P_1 = 0 + 0.4 = 0.4$

$q_3 = q_2 + P_2 = 0.4 + 0.3 = 0.7$

$q_4 = q_3 + P_3 = 0.7 + 0.2 = 0.9$

$q_5 = q_4 + P_4 = 0.9 + 0.1 = 1$

Step (5)

$q_1 = 0.00 \rightarrow l_1 = 2$

$q_2 = (0.4)_{10}$

$0.4 \times 2 = 0.8 \longrightarrow 0$

$0.8 \times 2 = 1.6 \longrightarrow 1$ $l_2 = 2$

$0.6 \times 2 = 1.2 \longrightarrow 1$

$0.2 \times 2 = 0.4 \longrightarrow 0$

$q_2 = (0.01)_2$

$q_3 = (0.7)_{10}$

$0.7 \times 2 = 1.4 \longrightarrow 1$

$0.4 \times 2 = 0.8 \longrightarrow 0$ $l_3 = 3$

$0.8 \times 2 = 1.6 \longrightarrow 1$

$q_3 = (0.101)_2$

$q_4 = (0.9)_{10}$

$0.9 \times 2 = 1.8 \longrightarrow 1$

$0.8 \times 2 = 1.6 \longrightarrow 1$

$0.6 \times 2 = 1.2 \longrightarrow 1$

$0.2 \times 2 = 0.4 \longrightarrow 0$

$l_4 = 4$

$q_4 = (0.1110)_2$

$q_5 = 1$

SYMBOLS	PROBABILITY	l_i	CODE WORD
D	0.4	2	00
C	0.3	2	01
B	0.2	3	101
A	0.1	4	1110

$$\eta = \frac{H(S)}{L} * 100$$

$$H(S) = \sum_{i=1}^{N} P_i \log_2\left(\frac{1}{P_i}\right)$$

$$= P_1 \log_2\left(\frac{1}{P_1}\right) + P_2 \log_2\left(\frac{1}{P_2}\right) + P_3 \log_2\left(\frac{1}{P_3}\right) + P_4 \log_2\left(\frac{1}{P_4}\right)$$

$$= 0.4 \log_2\left(\frac{1}{0.4}\right) + 0.3 \log_2\left(\frac{1}{0.3}\right) + 0.2 \log_2\left(\frac{1}{0.2}\right) +$$

$$0.1 \log_2\left(\frac{1}{0.1}\right)$$

$H(S) = 1.8464$ BPS

$$L = \sum_{i=1}^{N} P_i l_i$$

$$= P_1 l_1 + P_2 l_2 + P_3 l_3 + P_4 l_4$$

$$= 0.4 \times 2 + 0.3 \times 2 + 0.2 \times 3 + 0.1 \times 4$$

$$= 0.8 + 0.6 + 0.6 + 0.4$$

$$L = \underline{\underline{2.4}}$$

$$\eta_s = \frac{H(s)}{L} \times 100$$

$$\eta_s = \frac{1.8464}{2.4} \times 100$$

$$\eta_s = \underline{\underline{76.93\%}}$$

REDUNDANCY

$$R_s = 1 - \eta_s \quad [100 - \eta_s]$$

$$R_s = 23.06\%$$

*** Q2. Consider a discrete memoryless source with $S = (X, Y, Z)$ with corresponding probabilities $P = (0.5, 0.3, 0.2)$.

a] Find the code words for the symbols using SHANNON's ALGORITHM, also find the source efficiency & Redundancy.

b] Consider the 2nd extension of the source and find the code words, efficiency & Redundancy.

Ans:- a]

1. $S = (X, Y, Z)$

   $P = (0.5, 0.3, 0.2)$

2. $l_i \geq \log_2\left(\frac{1}{P_i}\right)$

   $l_1 \geq \log_2\left(\frac{1}{0.5}\right) = 1$

   $\boxed{l_1 = 1}$

30/07/2018

SHANNON – FANO ENCODING :-

This is an improvement over SHANNON's 1st Algorithm. It offers Better coding efficiency compared to SHANNON'S ALGORITHM.

PROCEDURE

1. Arrange the probability in Non-increasing order (decreasing order)
2. Group the probabilities into exactly 2 sets such that the sum of the probabilities in both the groups is almost equal. Assign bit '0' to all the elements of 1st group and bit '1' to all the elements of 2nd group.
3. Repeat step (2) by dividing each group into 2 sub-groups till no further division is possible.

(1.) Consider the following sources S = A, B, C, D, E, F and P = 0.1, 0.15, 0.25, 0.35, 0.08, 0.07.

Ans:-

(1.) P =

D	0.35
C	0.25
B	0.15
A	0.1
E	0.08
F	0.07

0.35	0
0.25	0
0.15	1
0.1	1
0.08	1
0.07	1

0.35	0
0.25	1

0.15	0
0.1	1
0.08	1
0.07	1

0.1	0
0.08	1
0.07	0

0.08	0
0.07	1

SYMBOLS	PROBABILITY	CODE WORD	l_i
D	0.35	00	2
C	0.25	01	2
B	0.15	10	2
A	0.1	110	3
E	0.08	1110	4
F	0.07	1101	4

$$\eta_S = \frac{H(s)}{L} * 100\%$$

$$H(s) = \sum_{i=1}^{N} P_i \log_2\left(\frac{1}{P_i}\right)$$

$$= P_1 \log_2\left(\frac{1}{P_1}\right) + P_2 \log_2\left(\frac{1}{P_2}\right) + P_3 \log_2\left(\frac{1}{P_3}\right) + P_4 \log_2\left(\frac{1}{P_4}\right) +$$

$$P_5 \log_2\left(\frac{1}{P_5}\right) + P_6 \log_2\left(\frac{1}{P_6}\right)$$

$$= \cancel{2.2037}\ 2.3329 \text{ BPS}$$

$$L = \sum_{i=1}^{N} P_i l_i$$

$$= P_1 l_1 + P_2 l_2 + P_3 l_3 + P_4 l_4 + P_5 l_5 + P_6 l_6$$

$$= \cancel{2.65}\ 2.4$$

$$\eta_s = \cancel{83.15\%}\ 97.20\%$$

$$R_s = 1 - \eta_s = \cancel{16.84\%}\ 2.8\%$$

② $S = A, B, C, D, E, F$

$P = 0.4, 0.2, 0.2, 0.1, 0.08, 0.02$

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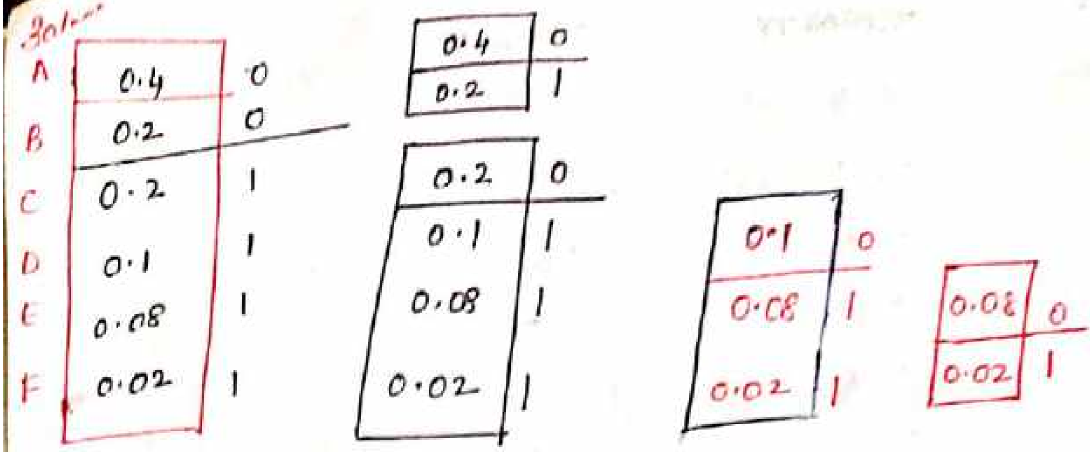
Soln



SYMBOLS	PROBABILITY	CODE WORD	l_i
A	0.4	00	2
B	0.2	01	2
C	0.2	10	2
D	0.1	110	3
E	0.08	1110	4
F	0.02	1111	4

$$H(s) = \sum_{i=1}^{N} P_i \log_2\left(\frac{1}{P_i}\right)$$

$$= 2.1941 \text{ BPS.}$$

$$L = \sum_{i=1}^{N} P_i l_i = \underline{\underline{2.3}}$$

$$= P_1 l_1 + P_2 l_2 + P_3 l_3 + P_4 l_4 + P_5 l_5 + P_6 l_6$$

$$\eta(s) = \frac{H(s)}{L} \times 100\%$$

$$= 95.39\%$$

$$R(\eta) = 1 - \eta_s = \underline{\underline{4.60\%}}$$

25/09/18

KRAFT Mc MILLAN INEQUALITY [KMI]

A Necessary and sufficient condition for existence of instantaneous code with code length $l_1, l_2, \ldots \ldots l_N$ is

$$\sum_{i=1}^{N} r^{-l_i} \leq 1$$

where $r \rightarrow$ Number of different symbols used in the code alphabet.

$N \rightarrow$ Total number of source symbols

Proof: The word length $l_1, l_2, \ldots \ldots l_N$ are arranged in the ascending order, so that $l_1 \leq l_2 \leq l_3 \leq \ldots \ldots \leq l_N$.

If we have to choose one length code words for all N number of source symbols, satisfying the prefix property, we can do so only when $N \leq r$.

If $N > r$, we have to go for combination of r symbols to form instantaneous code words.

Let n_i represent the number of messages encoded into code words of length i, then we have

for $i=1$, $n_1 \leq r$

for $i=2$, to get instantaneous code we must start encoding $(r-n_1)$ symbols only as the first digit and the second digit can be any of the r symbols of the code alphabet.

for $i=2$, $n_2 \leq (r-n_1)r$

$n_2 \leq r^2 - n_1 r$

for $i=3$, $n_3 \leq [(r-n_1)r - n_2]r$

$n_3 \leq (r^2 - n_1 r - n_2)r) \Rightarrow n_3 \leq r^3 - n_1 r^2 - n_2 r$

for length i, $n_i \leq r^i - n_1 r^{(i-1)} \ldots \ldots n_{(i-1)} r$.

rearranging,

$$n_i + n_1 r^{(i-1)} + n_2 r^{(i-2)} + \dots n_{(i-1)} r \le r^i$$

(46)

Xly both the sides by r^{-i}

$$r^{-i}\left(n_i + n_1 r^{(i-1)} + n_2 r^{(i-2)} + \dots n_{(i-1)} r\right) \le 1$$

$$\boxed{\sum_{K=1}^{i} n_K r^{-K} \le 1}$$

Since the actual number of messages has to be integer, we can rewrite the above equation as

(4) Consider a source $S = S_1, S_2$ with probabilities $P = 3/4, 1/4$ respectively. Obtain the SHANNON fanocode for the source S, its 2nd extension & 3rd extension. Calculate the efficiency for each case.

Ans

$S = s_1, s_2$

$P = 3/4, 1/4.$

$0.75, 0.25$

S_1	0.75	0
S_2	0.25	1

SYMBOLS	PROBABILITY	CODE WORD	LENGTH
S_1	0.75	0	1
S_2	0.25	1	1

$$H(S) = \sum_{i=1}^{N} p_i \log_2\left(\frac{1}{p_i}\right)$$

$$H(S) = P_1 \log_2\left(\frac{1}{P_1}\right) + P_2 \log_2\left(\frac{1}{P_2}\right)$$

$$\underline{H(S) = 0.8113 \text{ BPS}}$$

$$L = \sum_{i=1}^{N} p_i l_i$$

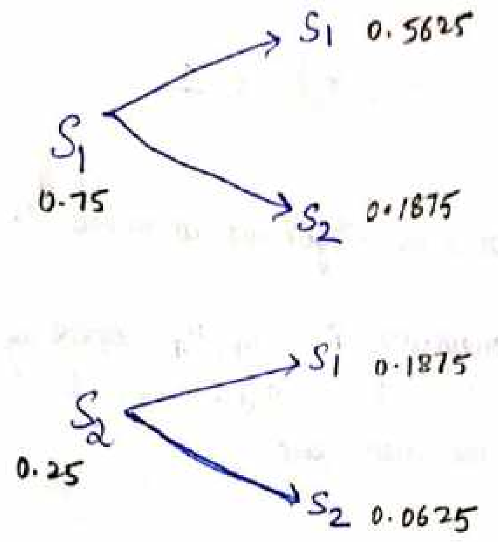
$$= P_1 l_1 + P_2 l_2$$

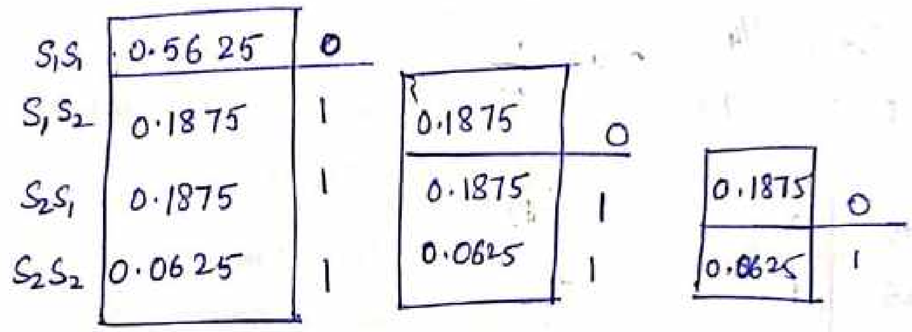
$$L = 1$$

$$\eta = \frac{H(S)}{L} \times 100\% = \underline{81.13\%}$$

$$R(s) = \underline{18.87\%}$$

(2) 2nd extension





SYMBOL	PROBABILITY	CODEWORD	LENGTH
S_1S_1	0.5625	0	1
S_1S_2	0.1875	1 0	2
S_2S_1	0.1875	1 1 0	3
S_2S_2	0.0625	1 1 1	3

$$H(S) = \sum_{i=1}^{N} p_i \log_2\left(\frac{1}{p_i}\right)$$

$$= P_1 \log_2\left(\frac{1}{P_1}\right) + P_2 \log_2\left(\frac{1}{P_2}\right) + P_3 \log_2\left(\frac{1}{P_3}\right) + P_4 \log_2\left(\frac{1}{P_4}\right)$$

$$H(S) = \underline{\underline{1.6226 \text{ BPS}}}$$

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$$L = \sum_{i=1}^{N} P_i l_i$$

$$= P_1 l_1 + P_2 l_2 + P_3 l_3 + P_4 l_4$$

$$L = 1.6875$$

$$\eta = \frac{H(S)}{L} \times 100\% = \underline{\underline{96.15\%}}$$

$$R(S) = 1 - \eta(S) = \underline{\underline{3.85\%}}$$

Case (iii) 3rd extension.

- $S_1 S_1$ (0.5625)
  - $S_1 S_1 S_1$ — $\frac{27}{64}$
  - $S_1 S_1 S_2$ — $9/64$
- $S_1 S_2$ (0.1875)
  - $S_1 S_2 S_1$ — $9/64$
  - $S_1 S_2 S_2$ — $3/64$
- $S_2 S_1$ (0.1875)
  - $S_2 S_1 S_1$ — $9/64$
  - $S_2 S_1 S_2$ — $3/64$
- $S_2 S_2$ (0.0625)
  - $S_2 S_2 S_1$ — $3/64$
  - $S_2 S_2 S_2$ — $1/64$

SYMBOLS

$S_1 S_1 S_1$	27/64
$S_1 S_1 S_2$	9/64
$S_1 S_2 S_1$	9/64
$S_2 S_1 S_1$	9/64
$S_1 S_2 S_2$	3/64
$S_2 S_1 S_2$	3/64
$S_2 S_2 S_1$	3/64
$S_2 S_2 S_2$	1/64

$S_1S_1S_1$	0.4219	0	0		
$S_1S_1S_2$	0.1406	0	1		
$S_1S_2S_1$	0.1406	1	0	0	
$S_2S_1S_1$	0.1406	1	0	1	
$S_1S_2S_2$	0.0469	1	1	0	0
$S_2S_1S_2$	0.0469	1	1	0	1
$S_2S_2S_1$	0.0469	1	1	1	0
$S_2S_2S_2$	0.0156	1	1	1	1

Symbols	probabilities	code word	length
$S_1S_1S_1$	0.4219	00	2
$S_1S_1S_2$	0.1406	01	2
$S_1S_2S_1$	0.1406	100	3
$S_2S_1S_1$	0.1406	101	3
$S_1S_2S_2$	0.0469	1100	4
$S_2S_1S_2$	0.0469	1101	4
$S_2S_2S_1$	0.0469	1110	4
$S_2S_2S_2$	0.0156	1111	4

$$H(S) = \sum_{i=1}^{N} P_i \log_2\left(\frac{1}{P_i}\right)$$

0·15

$$= P_1 \log_2\left(\frac{1}{P_1}\right) + P_2 \log_2\left(\frac{1}{P_2}\right) + P_3 \log_2\left(\frac{1}{P_3}\right) + P_4 \log_2\left(\frac{1}{P_4}\right)$$

$$+ P_5 \log_2\left(\frac{1}{P_5}\right) + P_6 \log_2\left(\frac{1}{P_6}\right) + P_7 \log_2\left(\frac{1}{P_7}\right) + P_8 \log_2\left(\frac{1}{P_8}\right)$$

$H(S) = \underline{2.4338 \text{ BPS}}$

$$L = \sum_{i=1}^{N} P_i l_i = P_1 l_1 + P_2 l_2 + P_3 l_3 + P_4 l_4 + P_5 l_5 + P_6 l_6 + P_7 l_7 + P_8 l_8$$

$L = 2.5938$

$\eta_{(s)} = \frac{H(s)}{L} * 100\% = 93.83\%$

$R(s) = 1 - \eta(s) = 6.17\%$

27/09/18

SHANNON'S FIRST THEOREM / FUNDAMENTAL THEOREM

(NOISE-LESS CODING THEOREM)

Shannon illustrated that the length l_i is related to the probability p_i given by

$$l_i = \log_2\left(\frac{1}{p_i}\right) \quad \text{— (1)}$$

The above equation signifies that larger the value of probability, smaller will be the length.

$$l_i \geq \log_2\left(\frac{1}{p_i}\right)$$

$$\log_2\left(\frac{1}{p_i}\right) \leq l_i \leq 1 + \log_2\left(\frac{1}{p_i}\right)$$

$$\frac{\log_2\left(\frac{1}{p_i}\right)}{\log_2 r} \leq l_i \leq 1 + \frac{\log_2\left(\frac{1}{p_i}\right)}{\log_2 r}$$

Multiplying both the sides by

$$\sum_{i=1}^{N} p_i$$

$$\sum_{i=1}^{N} p_i \frac{\log_2\left(\frac{1}{p_i}\right)}{\log_2 r} \leq \sum_{i=1}^{N} p_i l_i \leq \sum_{i=1}^{N} p_i + \sum_{i=1}^{N} p_i \frac{\log_2 \frac{1}{p_i}}{\log_2 r}$$

$$\frac{H(s)}{\log_2 r} \leq L \leq 1 + \frac{H(s)}{\log_2 r} \quad \text{— (2)}$$

$$H_r(s) \leq L \leq 1 + H_r(s) \text{ —— } (3)$$

$$\boxed{L \geq H_r(s)} \text{ —— } B$$

From eqn (2) for n^{th} extended source symbols we have,

$$\frac{H(s^n)}{\log_2 r} \leq L_n \leq 1 + \frac{H(s^n)}{\log_2 r}$$

$$\frac{n\,H(s)}{\log_2 r} \leq L_n \leq 1 + \frac{n\,H(s)}{\log_2 r}$$

dividing throughout by n:

$$\frac{H(s)}{\log_2 r} \leq \frac{L_n}{n} \leq \frac{1}{n} + \frac{H(s)}{\log_2 r}$$

$$\boxed{H_r(s) \leq \frac{L_n}{n} \leq \frac{1}{n} + H_r(s)} \text{ —— } (4)$$

Applying limits, we have

$$\boxed{\lim_{n \to \infty} \frac{L_n}{n} = H_r(s)}$$

where $L \to$ is the average length of code words for the basic source S.

It is true that, $L \geq \frac{L_n}{n}$

Eqn (4) is called as Noise-less coding Theorem / Shannon's First Theorem.

Noise-less coding Theorem states that "Given a code alphabet with r symbols and source alphabet of n symbols, the average length of code words can be made close to $H_r(s)$ by increasing the extension.

HUFFMANN CODE (COMPACT CODE)

These codes achieve minimum code word length among all other coding algorithms, hence they are called as COMPACT CODES.

PROCEDURE:-

The algorithm for encoding r-array codeword is as follows:-

Step ①:- Compute the number of stages (n) required for the encoding operation which is given by

$$n = \frac{N-r}{r-1} \quad , N \rightarrow \text{Total number of source symbols.}$$

Step ②:- The value of n has to be integer if not append minimum number of dummy symbols with zero probability so as to get n as an integer. However n will always take integer value for binary encoding.

Step ③:- Arrange the probabilities in descending order.

Step ④:- Combine the last 'r' probabilities in the set by summing up as a single probability and place the sum at the appropriate position in the set by re-ordering (either as high as possible or as low as possible). Now the set contains '$r-1$' elements less than the previous stage

Step ⑤:- Continue step ④ till we reach a position where we have only 'r' elements. Assign symbols $0, 1, \ldots\ldots r-1$ to those elements.

Step ⑥:- Traverse back by 1 stage. Find the 'r' probabilities in the stage whose sum is placed in the succeeding stage. Assign the code word to the sum as the prefix for the probabilities.

To differentiate, again assign $0, 1, \ldots\ldots r-1$ to those probabilities in front of their prefix.

Step ⑦:- Repeat step ⑥ till first stage is reached.

(1.) An information source produces a sequence of independent symbols $S = A, B, C, D, E, F, G$ with probabilities

$$P = \left(\tfrac{1}{3}, \tfrac{1}{27}, \tfrac{1}{3}, \tfrac{1}{9}, \tfrac{1}{9}, \tfrac{1}{27}, \tfrac{1}{27}\right)$$

Construct Binary & Ternary Code using HALFMANN Encoding procedure and also find its efficiency.

Soln:-

Step (1):- $n = \dfrac{N - r}{r - 1}$

$$n = \frac{7-2}{2-1} = \underline{5}$$

*- moving the sum as low as possible

Step (2):-

SYMBOL	PROBABILITY	CODE WORD	Stage 1		Stage 2		Stage 3		Stage 4		
A	1/3	1	1/3	1	1/3	1	1/3	1	1/3	1	2/3
C	1/3	00	1/3	00	1/3	00	1/3	00	1/3	00	1/3
D	1/9	011	1/9	011	1/9	011	2/9	010	1/3	01	
E	1/9	0100	1/9	0100	1/9	0100	1/9	011			
B	1/27	01011	2/27	01010	1/9	0101					
F	1/27	010100	1/27	01011							
G	1/27	010101									

SYMBOL	PROBABILITY	CODE WORD	LENGTH
A	1/3	1	1
C	1/3	00	2
D	1/9	011	3
E	1/9	0100	4
B	1/27	01011	5
F	1/27	010100	6
G	1/27	010101	6

50

1/3

$\eta = \frac{N-r}{r-1}$

$$H(S) = \sum_{i=1}^{N} p_i \log_2\left(\frac{1}{p_i}\right)$$

$$= \frac{1}{3}\log_2(3) + \frac{1}{3}\log_2(3) + \frac{1}{9}\log_2(9) + \frac{1}{9}\log_2(9) + \frac{1}{27}\log_2(27) + \frac{1}{27}\log_2(27) + \frac{1}{27}\log_2(27)$$

0
1

$H(S) = \underline{2.2894}$ BPS

$$L = \sum_{i=1}^{N} p_i l_i = 1/3 + 1/3\times2 + 1/9\times3 + 1/9\times4 + 1/27\times5 + 1/27\times6 + \frac{1}{27}\times6$$

$L = \underline{2.4074}$

$$\eta_s = \frac{H(s)}{L}\times100\%$$

$\eta_s = 95.09\%$

$R_{\eta} = 1 - \eta_s = \underline{4.91\%}$

Soln

Step ① $r = \frac{N-r}{r-1} = \frac{7-3}{3-1} = \frac{4}{2} = 2$

Step ②

SYMBOLS	PROBABILITIES		STAGE 1		STAGE 2	
A	1/3	0	1/3	0	1/3	0
C	1/3	1	1/3	1	1/3	1
D	1/9	20	1/9	20	1/3	2
E	1/9	21	1/9	21		
B	1/27	220	1/9	22		
F	1/27	221				
G	1/27	222				

SYMBOL	PROBABILITIES	CODEWORD	LENGTH
A	1/3	0	1
C	1/3	1	1
D	1/9	20	2
E	1/9	21	2
B	1/27	220	3
F	1/27	221	3
G	1/27	222	3

$L = \sum_{i=1}^{N} p_i l_i = 1.4444$

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$H(S) = 1.4444$ BPS

$$\eta_{(S)} = \frac{H(S)}{L} \times 100\%$$
$$=$$

$$H(S) = \sum_{i=1}^{N} p_i \log_2\left(\frac{1}{p_i}\right)$$

$$= 1.4444 \text{ BPS}$$

$$\eta_{(S)} = \frac{H(S)}{L} * 100\% = 100\% \quad (99.3\%).$$

$$R_{(S)} = 1 - \eta_S = 0\%$$

---

(2) Given $S = x, y, z$ and $P = 0.7, 0.15, 0.15$. $R = 2$. Form the compact code for the 1st & 2nd extension.

Ans:- $n = \frac{N - r}{r - 1} = \frac{3-2}{2-1} = 1$

Symbol	probabilities		Stage 1	
x	0.7	0	0.7	0
y	0.15	10	0.3	1
z	0.15	11		

Symbol	probability	codeword	length
X	0.7	0	1
Y	0.15	10	2
Z	0.15	11	2

$$H(S) = \sum_{i=1}^{N} p_i \log_2\left(\frac{1}{p_i}\right)$$

$$= 0.7 \log_2\left(\frac{1}{0.7}\right) + 2 * 0.15 \log_2\left(\frac{1}{0.15}\right)$$

Scanned with CamScanner

EXTENDED HAUFFMAN CODING

The procedure is same as Hauffman code with source extension.

As it can be seen from problem number (4) & (5), the codewords (moving as low as possible), the code words are same though the probabilities are different.

Also the lengths are equal. , that is Hauffmann Code is not unique if there is small variation in the probability.

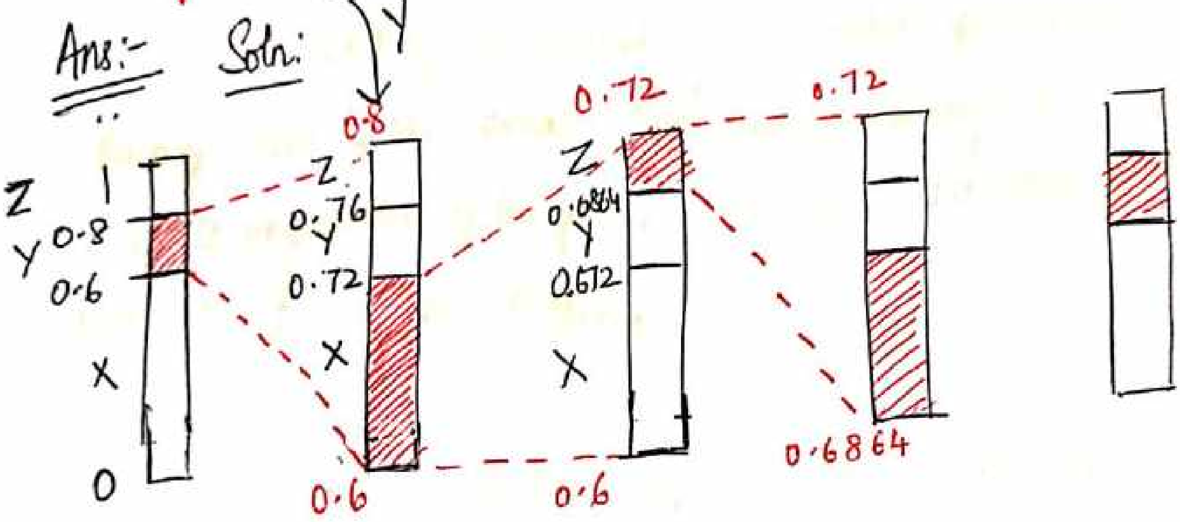
2) As it can be seen from problem number (6), the Hauffman code will be unique if the self information is an integer and equal to the length of the code., that is the probability should be negative power of 2 for the code to be UNIQUE.

ARITHMETIC CODING

The drawback of Hauffmann code can be overcome by using arithmetic coding.

(1) Consider a DMS with $S = \{X, Y, Z\}$ with respective probabilities $P = \{0.6, 0.2, 0.2\}$. Find the codeword for the message. YXZXY.

Ans:- Soln:



Value = difference * Prob + Base value

$= 0.2 * 0.6 + 0.6 = 0.72$
$= 0.2 * 0.8 + 0.6 = 0.76$ } Y

$= 0.12 * 0.6 + 0.6 = 0.672$
$= 0.12 * 0.72 + 0.6 = 0.6864$ } X

0.8

$\boxed{0.7024}$

25/9/17

→ Lempel - Ziv - Algorithm

Encode the following info using LZ alg:

THIS_IS_HIS_HIT

T, H, I, S, _, IS, _H, IS_, HI, T_

Dictionary location	Dictionary content	Fixed length code word
0001	T	0000T
0010	H	0000H
0011	I	0000I
0100	S	0000S
0101	_	0000_
0110	IS	0011S
0111	_H	0101H
1000	IS_	0110_
1001	HI	0010I
1010	T_	0001_

Since T is single & already written, add a _.

(0000, T) (codeword at locn 0, T)
(0000, H) (codeword at locn 0, H)
(0000, I) (codeword at locn 0, I)
(0000, S) (codeword at locn 0, S)
(0000, _) (codeword at locn 0, _)
(0011, S) (codeword at locn I, S)
(0101, H) (codeword at locn _, H)
(0110, _) (codeword at locn IS, _)
(0010, I) (codeword at locn H, I)
(0001, _) (codeword at locn T, _)

→ Encode using LZ alg, the string used is W_PNW_NW_WPN_P.

Soln W, _, P, N, W_, NW, _W, PN, _P.

Dict locn	Dict content	Fixed length code word	
0001	W	0000W	0000, W
0010	_	0000_	0000, _
0011	P	0000P	0000, P
0100	N	0000N	0000, N
0101	W_	0001_	0001, _
0110	NW	0100W	0100, W
0111	_W	0010W	0010, W
1000	PN	0011N	0011, N
1001	_P	0010P	0010, P

→ L-Z algorithm

The source coding algs usually require statistics of symbols. However in real time situations, the symbol probs would be unknown. Also most of the info sources used in real time possess memory. However Huffmann codes donot consider such dependency. ie Huffmann codes work very well for memoryless sources but fail to achieve optimal efficiency for sources with memory.

For sources with memory, better coding efficiency can be achieved by considering correlation among symbols in source coding. One such technique is L-Z algorithm (Lempel-Ziv).

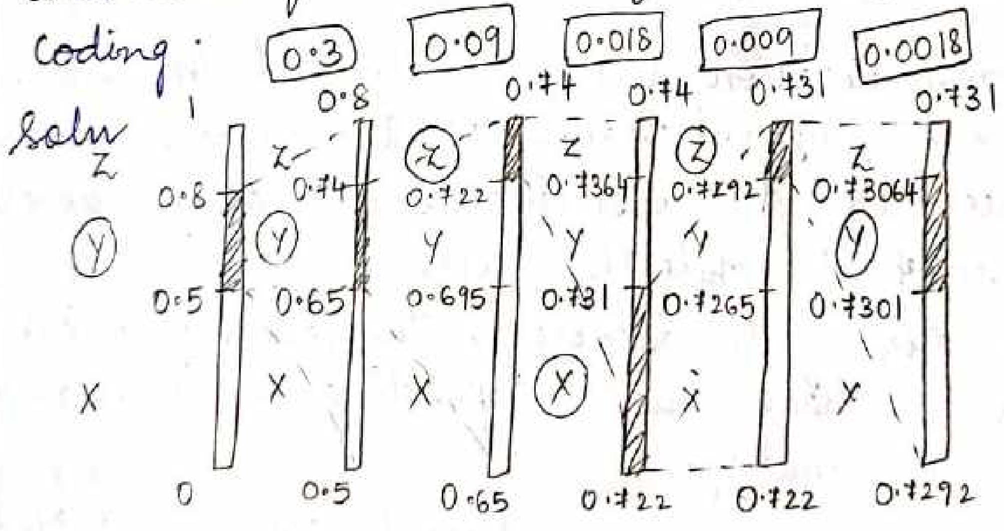
L-Z alg is a dictionary based approach in which a common dictionary is built at both sender

& receiver depending upon the String to be sent. & The codeword will be of the form (index, code) where index is the index of the dictionary where the longest match for the i/p sequence is found, code is the codeword for the symbol succeeding the longest match.

Index 0 is used if the symbol is encountered for the 1st time & not found in the dictionary. The dictionary is updated in accordance with the i/p string. Longer the string to be encoded better will be the compression achieved.

→ Consider a DMS with 3 symbols having respective probs, P = { 0·5, 0·3, 0·2 }. Find the code word for the message, YYZXZY using arithᶜ coding.

Soln



Hence the codeword / tag for YYZXZY is 0·7301.

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diff * Prob % + base value.

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Module 3

Information Channels

Communication channel

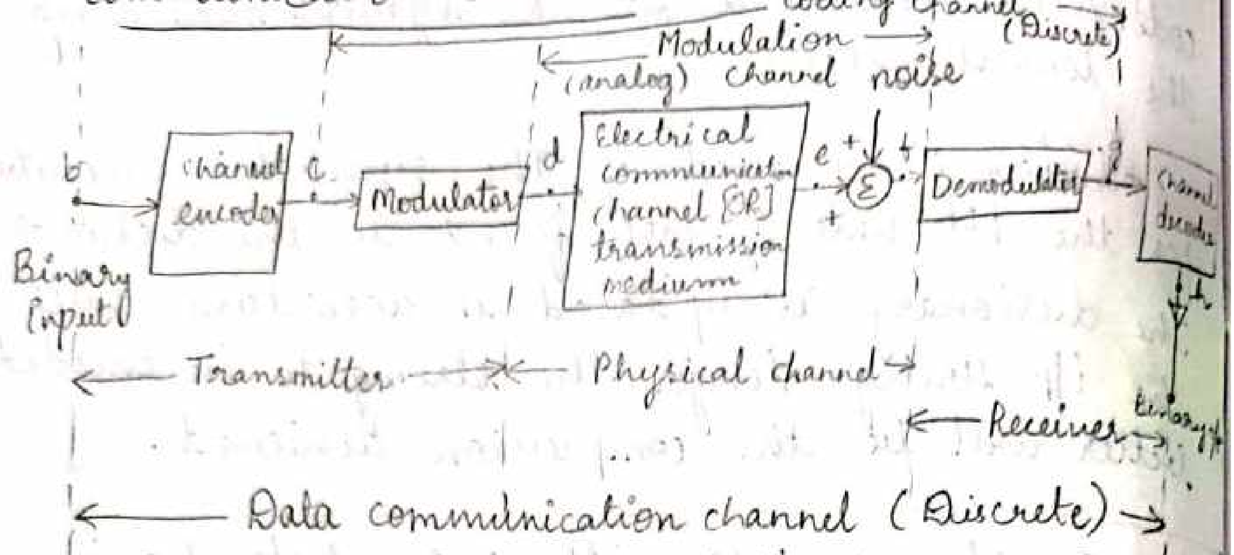


fig: Characterisation of binary communication Channel.

The communication system is divided into a trans-mitter, a physical channel or transmission medium & a receiver. A transmitter consists of an Encoder & a modulator while the receiver consists of a demodulator & a decoder. The communication channel is characterised depending on its terminal pts & functionality.

In the fig we have a discrete channel b/t pts c & g, Also refered as coding channel, This channel accepts a sequence of symbols at its input & produces a sequence of symbols at its output. This channel is characterised by a set of transition probabilities P_{ij} where P_{ij} is the channel probability for ith symbol.

These Probs will depend on the parameters of the modulator, transmission media, noise & demodulator. This dependence is transperant to a system designer who is concerned with the design of digital encoder & decoder.

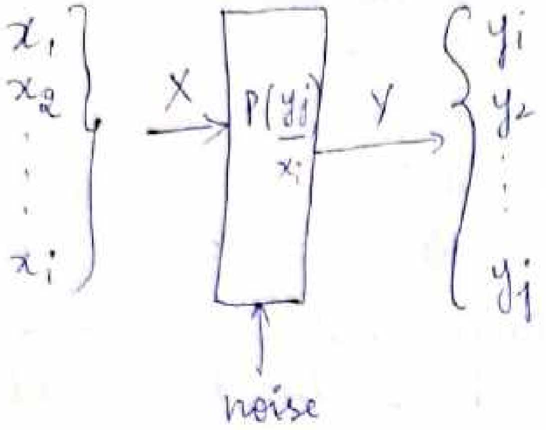
The communication channel b/t pts d & f in the system provides the electrical system b/t the transmitter & receiver. The i/p & o/p are analog electrical waveforms. This portion of channel is often called as continuous or modulation channel.
Ex: Analog Electrical communication channels such as voice band & white band telephone systems, high freq^ns radio system & troposcarrier system. These channels are subject to various disturbances, some are due to amplitude & frequency response variations of the channel within its passband. Other disturbances are due to channel characteristics with time & nonlinearities in the channel. All of these result in a channel modifying i/p signal in a deterministic fashion. In addition, the channel can also correct the signal statistically due to various types of additive & multiplicative noise. All of these introduce errors in data transmission & limit the max rate at which the data can be transferred over the channel.

→ Channel Models

The discrete communication channel can be modelled by representing them by a set of i/p alphabets $X = \{x_1, x_2, \ldots x_i\}$ & a set of o/p alphabets $Y = \{y_1, y_2 \ldots, y_j\}$.

A DM channel is completely described by a set of transition prob $P(y_j/x_i)$ where x_i denotes i/p symbols & y_j o/p symbol.

P_{yj} given $P(y_j/x_i)$ denotes the probability
denotes prob of receiving symbol y_j given that x_i was sent.



Channel Matrix [Prob transition matrix]

$$
\begin{array}{c} \\ \left[P(y_j/x_i) \right] = \end{array}
\begin{array}{c|cccc|}
 & Y_1 & Y_2 & \cdots & Y_j \\
x_1 & P(Y_1/x_1) & P(Y_2/x_1) & \cdots & P(Y_j/x_1) \\
x_2 & P(Y_1/x_2) & P(Y_2/x_2) & \cdots & P(Y_j/x_2) \\
\vdots & \vdots & \vdots & & \vdots \\
x_i & P(Y_1/x_i) & P(Y_2/x_i) & \cdots & P(Y_j/x_i)
\end{array}
$$

The conditional prob $P(y_j/x_i)$ is defined as the channel transition prob. These conditional probs can be represented in the form of a matrix with all i/p symbol represented row wise & the o/p symbols column wise. Such a matrix is known as Prob transition matrix (PTM) or channel matrix or noise matrix.

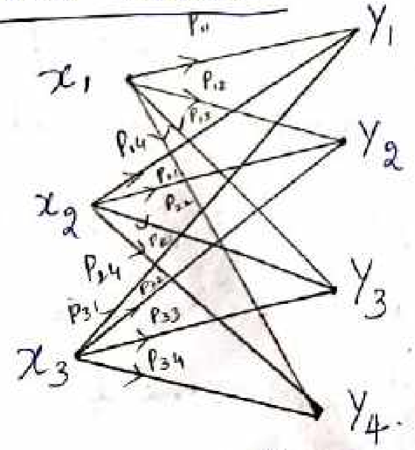


fig: Channel Diagram

Let us consider a discrete memoryless source 4 having an i/p alphabet $X = \{x_1, x_2, x_3\}$ and o/p alphabet $Y = \{y_1, y_2, y_3, y_4\}$. The different transition probs are marked in the chann diagram as shown above.

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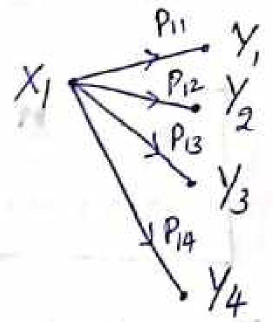
P_{11} is the prob that y_1 will be received when x_1 is transmitted.

ie $P_{11} = P(y_1/x_1)$; $P_{12} = P(y_2/x_1)$, $P_{13} = P(y_3/x_1)$

$P_{14} = P(y_4|x_1)$



$$[P(Y/X)] = \begin{array}{c c} & \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{21} & P_{22} & P_{23} & P_{24} \\ P_{31} & P_{32} & P_{33} & P_{34} \end{bmatrix} \end{array}$$

Since each i/p to the channel results in some outputs.

$$P_{11} + P_{12} + P_{13} + P_{14} = 1$$

$$\boxed{\sum_{j=1}^{m} P(y_j/x_i) = 1} \quad \forall i$$

WKT $P(AB) = P(B/A)\, P(A)$

$$P(x_i y_j) = P(y_j/x_i)\, P(x_i)$$

Here $P(x_i, y_j)$ is the joint Probability of x_i & y_j. If we add all the joint Probs for fixed y_j. then we get $\boxed{\sum_{i=1}^{n} P(x_i, y_j) = P(y_j)}$ $\forall j$

|||ly $\sum_{j=1}^{m} P(x_i, y_j) = P(x_i)$

Suppose that j^{th} symbol is received, i^{th} symbol is transmitted then error occurs.

The prob of error is given by

$$P_{e_{x_1}} = P(Y_2) + P(Y_3) + P(Y_4)$$

$$P_e = \sum_{\substack{j=1 \\ i \neq j}}^{M} P(Y_j)$$

$$P_e = \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i \neq j}}^{M} P(X_i\ Y_j)$$

$$\boxed{P_e = \sum_{i=1}^{M} \sum_{\substack{j=1 \\ i \neq j}}^{m} P(Y_j / X_i) \cdot P(X_i)}$$

The Prob of correct reception $= P_c = 1 - P_e$.

→ Joint Probability Matrix (JPM)

WKT X_i are the i/ps & Y_j are the o/ps then the Joint Prob is given by.

$$P(X_i, Y_j) = P(Y_j/X_i) \cdot P(X_i)$$

We known, the PTM,

$$P(Y_j / X_i) = \begin{bmatrix} P(Y_1/X_1) & P(Y_2/X_1) & \cdots & P(Y_m/X_1) \\ P(Y_1/X_2) & P(Y_2/X_2) & \cdots & \\ \vdots & & & \\ P(Y_1/X_n) & \cdots & \cdots & P(Y_m/X_n) \end{bmatrix}$$

Multiplying bs with $P(X_i)$

$$[P(Y_j/X_i)] \cdot P(X_i) = \begin{bmatrix} P(Y_1/X_1) \cdot P(X_1) & \cdots \\ P(Y_1/X_2) P(X_2) & \\ \vdots & \\ P(Y_1/X_n) \cdot P(X_n) & \end{bmatrix}$$

$$P(x_i, y_j) = \begin{bmatrix} P(x_1, y_1) & P(x_1, y_2) & & P(x_1, y_m) \\ P(x_2, y_1) & P(x_2, y_2) & & P(x_2, y_m) \\ \vdots & & & \\ P(x_n, y_1) & \cdots & & P(x_n, y_m) \end{bmatrix}$$ 5

The above matrix is the JPM [Joint probability matrix] which has the following Properties

(1) $P(y_j) = \sum_{i=1}^{n} P(x_i, y_j)$ ie.

ie the sum of all the elements of j^{th} column of JPM gives the prob of the j^{th} output. – Prob of all o/p symbols.

(2) $P(x_i)$ = summation ie $\sum_{j=1}^{m} P(x_i, y_j)$. $\forall\, i$ & $1 < i < n$

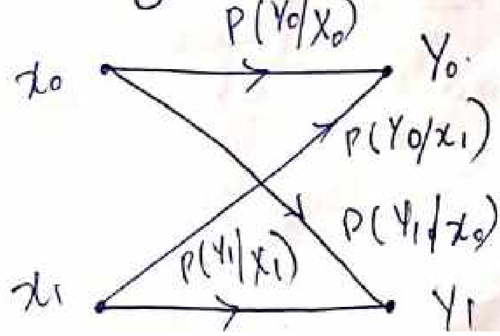
The prob of i/p can be obtained by adding all the elements of i^{th} row of JPM.

(3) $\sum_{i=1}^{n} P(x_i) = 1$

$\sum_{i=1}^{n} \sum_{j=1}^{m} [P(x_i, y_j)] = 1.$

ie the sum of all the elements of JPM = unity.

→ Binary Symmetric Channel.

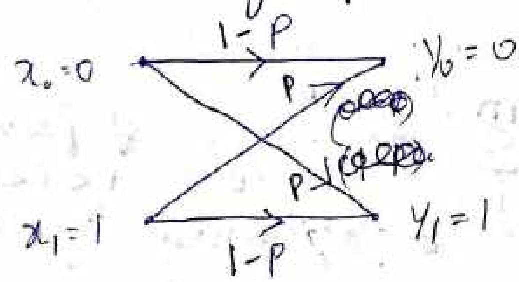


Let us consider a binary info channel where there are 2 i/p symbol $x(0)$ & $x(1)$ [x_0 & x_1] & 2 o/p symbols $y(0)$ & y_1 [y_0 & y_1]. The o/p symbol probs is given by $P(y_0) = P(y_0/x_0)\cdot P(x_0) + P(y_0/x_1)\,P(x_1)$

$|||^{ly}$ $P(y_1) = P(y_1/x_0)P(x_0) + P(y_1/x_1)$

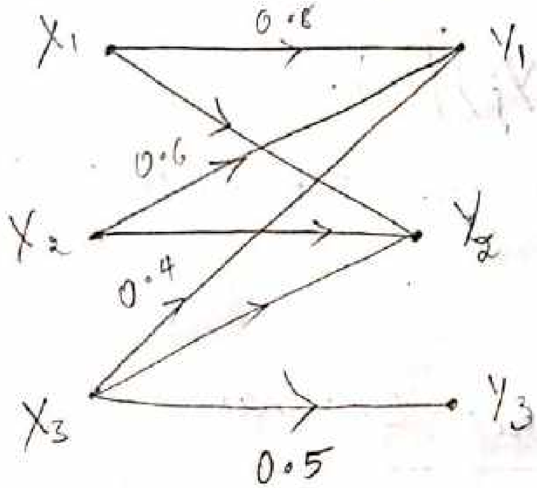
$$\begin{bmatrix} P(y_0) \\ P(y_1) \end{bmatrix} = [\,P(x_0) \cdot P(x_1)\,] \cdot \begin{bmatrix} P(y_0/x_0) & P(y_1/x_0) \\ P(y_0/x_1) & P(y_1/x_1) \end{bmatrix}$$

The binary symmetric channel is given by



$$P\begin{bmatrix} y_0 \\ y_1 \end{bmatrix} = [\,P(x_0) \;\; P(x_1)\,] \begin{bmatrix} 1-P & P \\ P & 1-P \end{bmatrix}$$

→ For the channel diagram shown below, write the corresponding channel matrix.



$$P[\,y_j/x_i\,] = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix}$$

$$\begin{bmatrix} 0.8 & 0.2 & 0 \\ 0.6 & 0.4 & 0 \\ 0.4 & 0.1 & 0.5 \end{bmatrix}$$

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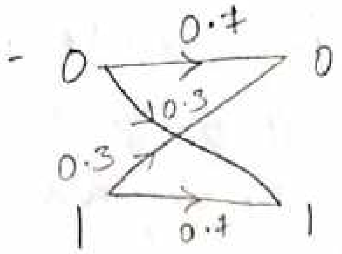
→ For the BSC shown in the fig, write the channel matrix. 6



$$\begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix}$$

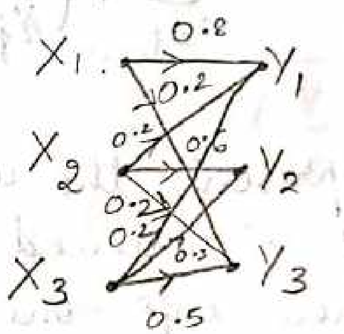
$P(x_1)$

→ For the given channel matrix, Finding the missing entries. & draw the channel diagram.

Soln. Channel matrix $= P(Y/X) = \begin{bmatrix} 0.8 & * & 0.2 \\ * & 0.6 & 0.2 \\ 0.2 & 0.3 & * \end{bmatrix}$

row addition $= 1$

$$\begin{bmatrix} 0.8 & 0 & 0.2 \\ 0.2 & 0.6 & 0.2 \\ 0.2 & 0.3 & 0.5 \end{bmatrix}$$



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System Entropies.

Input : X → System (channel) → output : Y

$H(X)$ (Input entropy) $H(Y)$ (Output entropy)

The i/p X & the o/p Y are sources with their own entropies H(X) & H(Y) as shown in figure where $H(X) = \sum_{i=1}^{n} P(X_i) \log_2 \frac{1}{P(X_i)}$

$|||^{ly}$ $H(Y) = \sum_{j=1}^{m} P(X_j) \log_2 \frac{1}{P(Y_j)}$

These source entropies ie $H(X)$ & $H(Y)$ represent the avg amount of info going into and coming out of the system per symbol. This avg uncertainity about the i/p & o/p is $H(X)$ & $H(Y)$. The entropy of the i/p symbols before transmission is known as Priori Entropy.

The entropy of the i/p symbols after transmission & reception of particular symbol say Y_j is defn as Posteriori (or) Conditional Entropy, which is denoted by $H\left(\frac{X}{Y_j}\right)$.

$$H\left(\frac{X}{Y_j}\right) = \sum_{i} P\left(\frac{X_i}{Y_j}\right) \log_2 \frac{1}{P\left(\frac{X_i}{Y_j}\right)}$$

This represents the uncertainity about X given that Y_j is received or equivalently how much more info one would gain by knowing X.

Averaging over all Y_j, we get Equivocation of X wrt Y, denoted by $H\left(\frac{X}{Y}\right)$.

$$H\left(\frac{X}{Y}\right) = E\left[H\left(\frac{X}{Y_j}\right)\right] \cancel{= E\left[\sum_i P\left(\frac{X_i}{Y_j}\right.\right.}$$

$$= \sum_{j=1}^{m} P(Y_j)\, H\left(\frac{X}{Y_j}\right)$$

$$= \sum_{j=1}^{m} P(Y_j) \cdot \sum_{i} P\left(\frac{X_i}{Y_j}\right) \log_2 \frac{1}{P\left(\frac{X_i}{Y_j}\right)}$$

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$$\boxed{H\left(\frac{X}{Y}\right) = \sum_j \sum_i P(X_i, Y_j) \log_2 \frac{1}{P\left[\frac{X_i}{Y_j}\right]}}$$ bits/symbol

$H\left(\frac{X}{Y}\right)$ ie equivocation specifies the receiver's avg uncertainity about X, when receiving Y. ie It specifies the avg uncertainity about the i/p of the channel, when channel output is ignored. It represents the amount of info lost due to noise. [Inter symbol Interference (ISI). |||ly if X_i is sent then the uncertainity about Y is the conditional entropy. $H(Y/X)$.

$$H\left(\frac{Y}{X}\right) = \sum_j \sum_i P(Y_j, X_i) \log_2 \frac{1}{P\left(\frac{Y_j}{X_i}\right)}$$

An observer trying to guess both the i/p of the channel X & its o/p Y will have avg uncertainity given by the joint entropy.

$$H(X, Y) = \sum_i \sum_j P(X_i, Y_j) \log_2 \frac{1}{P(X_i, Y_j)}$$

Problems *:

→ ① Prove that $H(X, Y) = H\left(\frac{X}{Y}\right) + H(Y)$.

Soln $H(X, Y) = \sum_i \sum_j P(X_i, Y_j) \log_2 \frac{1}{P(X_i, Y_j)}$

$$= \sum_i \sum_j P(Y_j) P\left(\frac{X_i}{Y_j}\right) \log_2 \frac{1}{P(Y_j) P\left(\frac{X_i}{Y_j}\right)}$$

$$= \sum_i \sum_j P(y_j)\, P\left(\frac{x_i}{y_j}\right)\left[\log_2 \frac{1}{P(y_j)} + \log_2 \frac{1}{P\left(\frac{x_i}{y_j}\right)}\right]$$

$$= \sum_i \sum_j P(y_j)\, P\left(\frac{x_i}{y_j}\right) \log_2 \frac{1}{P\left(\frac{x_i}{y_j}\right)} + \sum_j P(y_j) \log_2 \frac{1}{P(y_j)} \qquad \sum_{i=1}^{n} P\left(\frac{x_i}{y_j}\right) = 1$$

$$= \sum_i \sum_j P(x_i, y_j) \log_2 \frac{1}{P\left(\frac{x_i}{y_j}\right)} + H(Y)$$

$$= H\left(\frac{X}{Y}\right) + H(Y) \qquad \text{lly } H(X, Y) = H(Y/X) + H(X)$$

$$\boxed{P(A,B) = P(B)\, P\left(\frac{A}{B}\right) = P(A)\, P\left(\frac{B}{A}\right)}$$

→ (2) For the JPM matrix given Find $H(X)$, $H(Y/X)$, $H(X/Y)$, $H(X, Y)$, ~~$H(Y/X)$~~, $H(Y)$

$$P(X, Y) = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{bmatrix} 0.05 & 0 & 0.2 & 0.05 \\ 0 & 0.1 & 0.1 & 0 \\ 0 & 0 & 0.2 & 0.1 \\ 0.05 & 0.05 & 0 & 0.1 \end{bmatrix}$$

Soln $P(y_j) = \sum_{i=1}^{m} P(x_i, y_j)$

$P(x_i) = \sum_{j=1}^{m} P(x_i, y_j)$

$P(Y_1) = 0.1$; $P(Y_2) = 0.15$; $P(Y_3) = 0.5$, $P(Y_4) = 0.2$ 8

$$H(Y) = 0.1 \log_2 \frac{1}{0.1} + 0.15 \log_2 \frac{1}{0.15} + \dots$$

$H(Y) = 1.7427$ bps

$$H(X) = \sum_{i=1}^{n} P(X_i) \log_2 \frac{1}{P(X_i)}$$

$$= 0.3 \log_2 \frac{1}{0.3} \times 2 + 0.2 \log_2 \frac{1}{0.2} \times 2$$

$$= 1.9710$$

$$H(X, Y) = \sum_i \sum_j P(X_i Y_j) \log_2 \frac{1}{P(X_i Y_j)}$$

$$= 4 \times 0.5 \log_2 \frac{1}{0.5} + 4 \times 0.1 \log_2 \frac{1}{0.1} + 2 \times 0.2 \log_2 \frac{1}{0.2} = 3.122 \text{ bps.}$$

$$H(X, Y) = H\left(\frac{X}{Y}\right) + H(Y)$$

$$H\left(\frac{X}{Y}\right) = 3.122 - 1.7427 = 1.3793$$

$$H\left(\frac{Y}{X}\right) = 3.1220 - 1.9710 = 1.151 \text{ bps}$$

$H(X, Y) - H(X)$.

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→ A channel matrix is given as follows:

$$P(Y/X) = \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.2 & 0.6 & 0.2 \\ 0.2 & 0.2 & 0.6 \end{bmatrix}$$

Also i/p probs $P(X) = [1/3 \quad 1/3 \quad 1/3]$ Find $H(X)$, $H(Y)$, $H(X, Y)$, $H(Y/X)$ & $H(X/Y)$.

Soln $P(X) \cdot P(Y/X) = \begin{bmatrix} 1/3 & 1/3 & 1/3 \end{bmatrix} \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.2 & 0.6 & 0.2 \\ 0.2 & 0.2 & 0.6 \end{bmatrix}$

$\cancel{= [1/3 \quad 1/3 \quad 1/3]}$

$\cancel{P(Y_j) = [1/3 \quad 1/3 \quad 1/3]}$ $\quad P(Y/X) \cdot P(X) = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} [\cdot \cdot]$

$\cancel{P(X_i) = [1]}$

$$P(X, Y) = \begin{bmatrix} 0.2 & 0.066 & 0.066 \\ 0.066 & 0.2 & 0.066 \\ 0.066 & 0.066 & 0.2 \end{bmatrix}$$

$$P(X_i) = \begin{bmatrix} 0.332 \\ 0.332 \\ 0.332 \end{bmatrix} \qquad P(Y_j) = \begin{bmatrix} 0.332 & 0.332 & 0.332 \end{bmatrix}$$

$$H(X) = \sum_{i=1}^{n} P(X_i) \log_2 \frac{1}{P(X_i)}$$

$$H(X) = 3 \times 0.332 \log_2 \frac{1}{0.332} = 1.584 \text{ bps}$$

$$H(Y) = \sum_{j=1}^{m} P(Y_j) \log_2 \frac{1}{P(Y_j)}$$

$$H(Y) = 3 \times 0.332 \log_2 \frac{1}{0.332} = 1.584 \text{ bps}$$

$$H(X, Y) = 0.2 \times 3 \log_2 \frac{1}{0.2} + 6 \times 0.066 \log_2 \frac{1}{0.066}$$

$$H(X, Y) = 2.95 \text{ bps}$$

$$H(Y/X) = H(X, Y) - H(X) = 2.95 - 1.584$$

$H(Y/X) = 1.366$ bps.

$H(X/Y) = H(X,Y) - H(Y)$

$= 2.95 - 1.584 = 1.366$ bps.

→ For the JPM matrix given, $P(X,Y)$ matrix

$$\begin{bmatrix} 0.15 & 0 & 0 & 0.05 \\ 0 & 0.2 & 0.15 & 0 \\ 0 & 0.05 & 0.15 & 0.05 \\ 0.1 & 0.1 & 0.2 & 0.1 \end{bmatrix}$$

Find $H(X)$, $H(Y)$, $H(X/Y)$, $H(Y/X)$, $H(X,Y)$.

Soln $P(X_i) = \begin{bmatrix} 0.2 \\ 0.35 \\ 0.15 \\ 0.2 \end{bmatrix}$ $P(Y_j) = [0.25 \; 0.3 \; 0.25 \; 0.2]$

$$H(X) = \sum_{i=1}^{n} P(X_i) \log_2 \frac{1}{P(X_i)}$$

$$= 0.2 \log_2 \frac{1}{0.2} + 0.15 \log_2 \frac{1}{0.15} + 0.35 \log_2 \frac{1}{0.35}$$

$$+ 0.2 \log_2 \frac{1}{0.2}$$

$= 0.52 + 0.41 + 0.53 + 0.46.$

$= 1.92$ bps.

$$H(Y) = \sum_{j=1}^{m} P(Y_j) \log_2 \frac{1}{P(Y_j)}$$

$$= 0.25 \times 2 \log_2 \frac{1}{0.25} + 0.2 \log_2 \frac{1}{0.2} + 0.3 \log_2 \frac{1}{0.3}$$

$= 1.98$ bps.

$$H(X,Y) = 2 \times 0.15 \log_2 \frac{1}{0.15} + 4 \times 0.1 \log_2 \frac{1}{0.1} +$$

$$2 \times 0.05 \log_2 \frac{1}{0.05} + 0.2 \log_2 \frac{1}{0.2}$$

$$= 0.82 + 1.328 + 0.432 + 0.46$$

$$= 3.04 \text{ bps.}$$

$$H(Y/X) = H(X,Y) - H(X)$$

$$= 3.04 - 1.92$$

$$= 1.12 \text{ bps.}$$

$$H(X/Y) = 3.04 - 1.98$$

$$= 1.06 \text{ bps.}$$

→ A channel matrix is given as $P(Y/X)$ is

$$\begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.3 & 0.2 & 0.5 \\ 0.1 & 0.4 & 0.5 \end{bmatrix}, \quad P(X) = \left[\tfrac{1}{2} \;\; \tfrac{1}{4} \;\; \tfrac{1}{4}\right].$$

Find $H(X)$, $H(Y)$, $H(X,Y)$, $H(X/Y)$, $H(Y/X)$.

Soln

$$P(X,Y) = \begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.3 & 0.2 & 0.5 \\ 0.1 & 0.4 & 0.5 \end{bmatrix} \left[\tfrac{1}{2} \;\; \tfrac{1}{4} \;\; \tfrac{1}{4}\right]$$

$$P(X,Y) = \begin{bmatrix} 0.2 & 0.15 & 0.15 \\ 0.075 & 0.05 & 0.125 \\ 0.025 & 0.1 & 0.125 \end{bmatrix}$$

$$P(X_i) = \begin{bmatrix} 0.5 \\ 0.25 \\ 0.25 \end{bmatrix}; \quad P(Y_j) = \left[0.3 \;\; 0.3 \;\; 0.4\right]$$

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$$H(X) = \sum_{i=1}^{n} P(X_i) \log_2 \frac{1}{P(X_i)}$$

$$= 0.5 \log_2 \frac{1}{0.5} + 0.25 \log_2 \frac{1}{0.25} + 0.275 \log_2 \frac{1}{0.275}$$

$$= 0.5 + 0.5 + 0.500$$

$$= 1.500 \text{ bps.}$$

$$H(Y) = \sum_{j=1}^{m} P(Y_j) \log_2 \frac{1}{P(Y_j)}.$$

$$= 0.300 \log_2 \frac{1}{0.300} + 0.3 \log_2 \frac{1}{0.3} + 0.4 \log_2 \frac{1}{0.4}$$

$$= 1.570 \text{ bps.} \qquad [0.52 * 2 + 0.53]$$

$$H(X, Y) = 3.08 \text{ bps.}$$

$$H(Y/X) = H(X, Y) - H(X)$$

$$= 1.567 \text{ bps.}$$

$$H(X/Y) = H(X, Y) - H(Y)$$

$$= 1.50 \text{ bps.}$$

29/10/17 → Mutual Information (Transinformation)

WKT $H(X) = \sum_{i=1}^{N} P(X_i) \log_2 \frac{1}{P(X_i)}$

$$H(X/Y) = \sum_i \sum_j P(X_i, Y_j) \log_2 \frac{1}{P(\frac{X_i}{Y_j})}$$

H(X) is the transmitted info over the channel, an avg amount of info = equivocation $H(X/Y)$ is lost in the channel due to the channel noise. The remaining info received at the receiver with respect to the observed o/p symbol is referred to as Mutual info defined by $I(X;Y)$.

$$\boxed{I(X;Y) = H(X) - H\left(\frac{X}{Y}\right)}$$

$$I(X;Y) = \sum_i P(X_i)\log_2 \frac{1}{P(X_i)} - \sum_i\sum_j P(X_i,Y_j)\log_2 \frac{1}{P(X_i/Y_j)}$$

WKT $\sum_j P\left(\frac{Y_j}{X_i}\right) = 1$. $\sum_j P\left(\frac{Y_j}{X_i}\right) = 1$

$$\Rightarrow I(X;Y) = \sum_i\sum_j P(X_i)\, P(Y_j/X_i)\log_2 \frac{1}{P(X_i)} -$$

$$\sum_i\sum_j P(X_i,Y_j)\log_2 \frac{1}{P\left(\frac{X_i}{Y_j}\right)}$$

$$= \sum_i\sum_j P(X_i,Y_j)\log_2 \frac{1}{P(X_i)} - \sum_i\sum_j P(X_i,Y_j)\log_2 \frac{1}{P\left(\frac{X_i}{Y_j}\right)}$$

$$= \sum_i\sum_j P(X_i,Y_j)\left[\log_2 \frac{1}{P(X_i)} - \log_2 \frac{1}{P\left(\frac{X_i}{Y_j}\right)}\right]$$

$$= \sum_i\sum_j P(X_i,Y_j)\left[\log_2 \frac{P(X_i/Y_j)}{P(X_i)}\right].$$

$$= \sum_i\sum_j P(X_i,Y_j)\log_2\left[\frac{P(X_i,Y_j)}{P(X_i)P(Y_j)}\right]$$

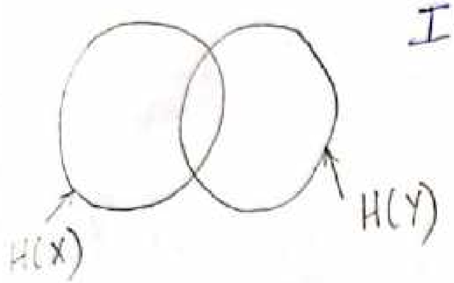
ie $$\boxed{I(X;Y) = \sum_i\sum_j P(X_i,Y_j)\log_2 \frac{P(X_i,Y_j)}{P(X_i)P(Y_j)}}$$

$|||^{ly}$ $$I(Y;X) = \sum_i\sum_j P(X_i,Y_j)\log_2 \frac{P(X_i,Y_j)}{P(X_i)P(Y_j)}$$

ie $I(X;Y) = I(Y;X)$

$$I(Y;X) = H(Y) - H(Y/X).$$

→ Graphical Representation of Mutual Info



$$I(X;Y) = H(X) - H\left(\frac{X}{Y}\right)$$

ie

$I(X;Y)$ can be thought of as intersection of sets corresponding to $|X \cap Y| = |Y| - |Y/X|$

$$H(X,Y) = H(X) + H(Y/X)$$

$|||^{ly}$ $H(Y,X) = H(X,Y) = H(Y) + H(X/Y)$

$$H(X) + H(Y/X) = H(Y) + H(X/Y)$$

$$H(X) - H(X/Y) = H(Y) - H(Y/X)$$

$$\boxed{I(X;Y) = I(Y;X)}$$

This says that the o/p tells exactly as much about the i/p as the i/p tells about the o/p.

$$H(X,Y) = H(Y) + H(X/Y)$$

also $H(X/Y) = H(X,Y) - H(Y)$

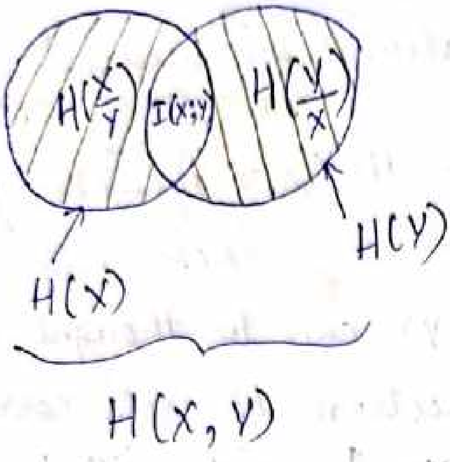
but $I(X;Y) = H(X) - H(X/Y)$

$$I(X;Y) = H(X) - H(X,Y) + H(Y)$$

which is analogous to : $|X \cap Y| = |X| + |Y| - |X \cup Y|$

All the above entropy relations can be represented graphically. The source entropy $H(X)$ & the destination entropy $H(Y)$ have been represented by 2 overlapping circles as shown in figure.

The overlap b/t the two circles shows $I(X;Y)$ which is the mutual info.



→ Properties

1. Mutual info of a channel exhibits symmetric property ie $I(X;Y) = I(Y;X)$.

2. The Mutual info is always non negative.

ie $I(X;Y) \geq 0$

3. Mutual info can be represented in terms of channel entropies ie

$$I(X;Y) = H(Y) - H(Y/X) = H(X) - H(X/Y).$$

4. Mutual info can be written in terms of Joint entropy. $I(X;Y) = H(X) + H(Y) - H(X,Y)$

→ PT $I(X;Y) \geq 0$.

Proof: $I(X;Y) = \sum_i \sum_j P(X_i, Y_j) \log_2 \left[\dfrac{P(X_i, Y_j)}{P(X_i) P(Y_j)} \right]$

From the inequality WKT

$$\ln \frac{1}{x} \geq 1 - x$$

$$I(X;Y) = \log_2 e \sum_i \sum_j P(X_i, Y_j) \log_e \left[\frac{P(X_i, Y_j)}{P(X_i) P(Y_j)} \right]$$ 12

Consider $\log_e \frac{P(X_i, Y_j)}{P(X_i) P(Y_j)} \geq 1 - \frac{P(X_i) P(Y_j)}{P(X_i, Y_j)} \longrightarrow (1)$

Multiplying both the sides by $\sum_i \sum_j P(X_i, Y_j)$

$$\sum_i \sum_j \ln \left[\frac{P(X_i, Y_j)}{P(X_i) P(Y_j)} \right] \cdot P(X_i, Y_j) \geq \sum_i \sum_j P(X_i, Y_j) \cdot \left[1 - \frac{P(X_i) P(Y_j)}{P(X_i, Y_j)} \right]$$

Multiplying b.s by $\log_2 e$.

$$\log_2 e \sum_i \sum_j P(X_i, Y_j) \ln \left[\frac{P(X_i, Y_j)}{P(X_i) P(Y_j)} \right] \geq$$

$$\log_2 e \sum_i \sum_j P(X_i, Y_j) \left[1 - \frac{P(X_i) P(Y_j)}{P(X_i, Y_j)} \right]$$

ie $\geq \log_2 e \left[\sum_i \sum_j \left[P(X_i, Y_j) - P(X_i) P(Y_j) \right] \right]$

ie $\geq \log_2 e \left[\sum_i \sum_j P(X_i, Y_j) - \sum_i P(X_i) \sum_j P(Y_j) \right]$

WKT $\sum_i \sum_j P(X_i, Y_j) = \sum_i P(X_i) = \sum_j P(Y_j) = 1$.

ie $\geq \log_2 e \left[1 - 1 \right]$

$I(X;Y) \geq 0$ ie $\boxed{I(X;Y) \geq 0}$

→ Average rate of information

WKT the rate of transmission from a source X is $R = H(X) \cdot r_s$ bits / second or bps.
where r_s is the symbol rate.

The avg. info rate is decided by how noisy the channel is & how much info can be reconstructed at the receiver.

As some amt. of info is lost in the noisy channel, the net amt of info at the receiver is the mutual info $I(X;Y)$.

Hence the avg rate of info transfer is given by : $R_{tr} = I(X;Y)\, r_s$ bps.

Suppose the channel is too noisy, it may so happen that the o/p of the channel may become statistically independent of the input.

ie $H\left(\frac{Y}{X}\right) = H(Y)$

WKT $I(X;Y) = H(Y) - H(Y/X)$

$= 0$

ie all the info sent into the channel is lost & no info. is transmitted over the channel, Hence zero info at the receiver. The channel is said to be useless.

→ Relative Entropy (Kullback - Leibler Distance)

Consider a discrete distribution that has a probability func^n P_K & another discrete dist^n that has a prob func^n q_K. Then we def^n the relative entropy of P wrt q given by

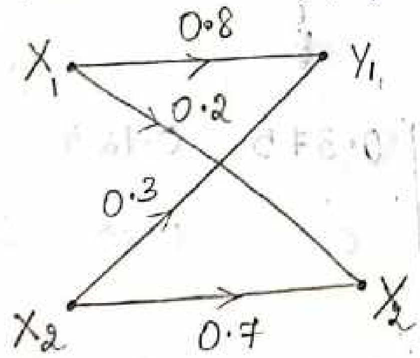
$$d = \sum P_K \log_2 \left(\frac{P_K}{q_K} \right)$$

$$d(P, q) \neq d(q, P)$$

Also, It is always non negative & will be equal to zero iff $P_K = q_K$.

Problems

① Find the mutual info about the channel shown in the fig given : $P(X_1) = 0.6$, $P(X_2) = 0.4$.



$I(X, Y) = ?$

Soln $P(Y/X) \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix}$ $P(X) = [0.6 \quad 0.4]$.

$$P(X,Y) = P(Y/X)\,P(X) \qquad P(Y) = [0.6 \quad 0.4]$$

$$= \begin{bmatrix} 0.48 & 0.12 \\ 0.12 & 0.28 \end{bmatrix}$$

$$H(X,Y) = 0.48 \log_2 \frac{1}{0.48} + 0.12 \log_2 \frac{1}{0.12} + 0.12 \log_2 \frac{1}{0.12} + 0.28 \log_2 \frac{1}{0.28}$$

$$= 0.5 + 0.367 + 0.367 + 0.514 = 1.756 \text{ bps}$$

$$H(X) = \sum_i P(x_i) \log_2 \frac{1}{P(x_i)}$$

$$= 0.4 \log_2 \frac{1}{0.4} + 0.6 \log_2 \frac{1}{0.6}$$

$H(X) = 0.97$ bps

$$I(X;Y) = H(X) + H(Y) - H(X,Y)$$

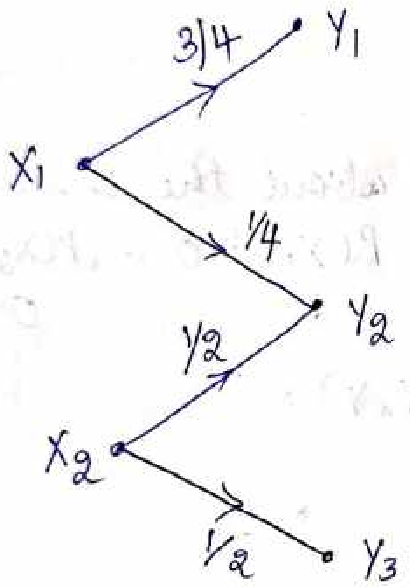
$$= 0.9709 + 0.9709 - 1.756$$

$$= 0.185 \text{ bps}$$

---

② $P(x_1) = P(x_2) = 0.5$; $R = ?$

$r_s = 10000$ sym/s



$$P(X) = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}$$

$$P(Y/X) = \begin{bmatrix} 3/4 & 1/4 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

$$P(X,Y) = \begin{bmatrix} 0.375 & 0.125 & 0 \\ 0 & 0.25 & 0.25 \end{bmatrix}$$

$$H(X,Y) = 0.375 \log_2 \frac{1}{0.375} + 2 \times 0.25 \log_2 \frac{1}{0.25} + 0.125 \log_2 \frac{1}{0.125}$$

$$= 1.9056 \text{ bps}.$$

$$H(X) = 2 \times 0.5 \log_2 \frac{1}{0.5} = 1 \text{ bps}.$$

$$H(Y) = 1.561278 \text{ bps}.$$

$$I(X,Y) = 1 + 1.561278 - 1.9056 = 0.6556 \text{ bps}$$

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$R = I(X, Y) \cdot r_s$

$= 0.655678 \times 10000$

$R = 6556.78$ bits / sec

---

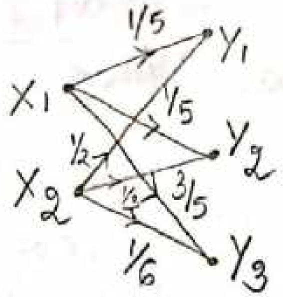
③ Two noisy channels are cascaded,

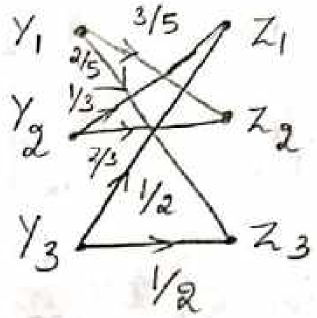
$$P(Y/X) = \begin{bmatrix} 1/5 & 1/5 & 3/5 \\ 1/2 & 1/3 & 1/6 \end{bmatrix} ; \quad P(Z/Y) = \begin{bmatrix} 0 & 3/5 & 2/5 \\ 1/3 & 2/3 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

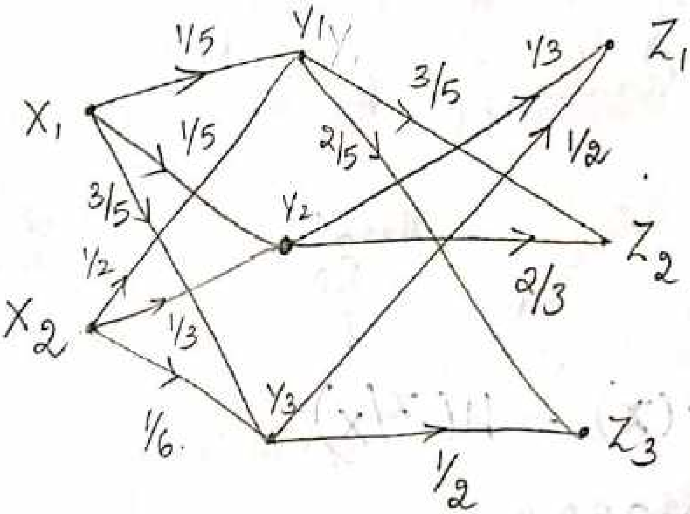
$P(X_1) = P(X_2) = 1/2$.

$I(X, Z) = ?$.

Soln.







$P(Z_1/X_1) = \frac{1}{5} \times \frac{1}{3} + \frac{3}{5} \times \frac{1}{2}$

$= 11/30$

$P(Z_1/X_2) = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9} +$

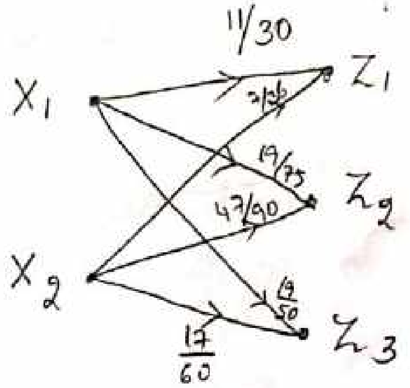
$\frac{1}{6} \times \frac{1}{2} = \frac{7}{36}$

$P(Z_2/X_1) = \frac{1}{5} \times \frac{3}{5} + \frac{1}{5} \times \frac{2}{3}$

$= 19/75$.

$P(Z_2/X_2) = \frac{1}{2} \times \frac{3}{5} + \frac{1}{3} \times \frac{2}{3}$

$= 47/90$



$P(Z_3/X_1) = \frac{2}{25} + \frac{3}{10}$

$= 19/50$

$P(Z_3/X_2) = \frac{2}{10} + \frac{1}{12}$

$= 17/60$

$$P(Z/X) = \begin{bmatrix} 11/30 & 19/75 & 19/50 \\ 7/36 & 47/90 & 17/60 \end{bmatrix}$$

$$P(X,Z) = \begin{bmatrix} 11/30 & 19/75 & 19/50 \\ 7/36 & 47/90 & 17/60 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \end{bmatrix}$$

$$P(X,Z) = \begin{bmatrix} 11/60 & 19/150 & 19/100 \\ 7/72 & 47/180 & 17/120 \end{bmatrix}$$

$$P(X_i) = \begin{bmatrix} 1/2 & 1/2 \end{bmatrix}$$

$$P(Z_i) = \begin{bmatrix} 101/360 & 349/900 & 199/600 \end{bmatrix}$$

$$H(X) = \sum_{i=1}^{M} P(X_i) \log_2 \frac{1}{P(X_i)} = 1 \text{ bps.}$$

$$H(Z) = \sum_{j=1}^{M} P(Y_j) \log_2 \frac{1}{P(Y_j)} = 1.57 \text{ bps}$$

$$I(X,Z) = H(\cancel{X}) - H(Z/X)$$

$$= 0.0588 \text{ bps.}$$

11/10/17

→ Capacity of discrete memoryless channel

A capacity of discrete memoryless noisy channel is defⁿ as the max possible rate of info transmission over the channel.

The max rate of transmission occurs if the source is matched to the channel.

ie $C = Max[R_t]$

$$C = Max\left[H(X) - H(X/Y)\right] r_s$$

→ Shannon's theorem on Channel capacity

(Shannon's 2nd theorem)

WKT, The rate of info transmission is given by $R_{tr} = \left[H(X) - H(X/Y)\right] r_s$.

& the channel capacity is given by:

$$C = Max\left[H(X) - H(X/Y)\right] r_s.$$

Shannon's theorem on channel capacity is given in 2 statements.

① Positive statement: Shannon's theorem on channel capacity states that "when the rate of info transmission $R_{tr} \leq C$, then there exists a coding technique which enables transmission over a channel with as small as probability of error as possible with even in the presence of noise in the channel.

This situation is similar to amplitude modulation with m as the modulation index. As long as $m \leq 1$, transmission of modulated signal is possible without errors. But when $m > 1$, transmission is possible but there will be errors introduced due to over modulation.

(2) Negative statement : If $R_{tr} > C$, then the reliable transmission of info is not possible ie it leads to errors. Thus the errors cannot be controlled by any coding technique & the probability of error of receiving the correct msg becomes close to unity.

→ Channel Efficiency & Redundancy

Channel efficiency is def^n as :

$$\eta_{ch} = \frac{R_{tr}}{C} \times 100\%$$

$$\eta_{ch} = \frac{[H(X) - H(X/Y)]}{Max[H(X) - H(X/Y)]} \times 100\% = \frac{I(X,Y) \times 100}{max(I(X,Y))}$$

Channel Redundancy : $R_{\eta_{ch}}$

$$R_{\eta_{ch}} = 1 - \eta_{ch}$$

→ Special channels

(1) Symmetric / uniform channel

(2) Binary symmetric channel (BSC)

(3) Binary Erasure channel (BEC)

(4) Noiseless channel

(5) Deterministic channel

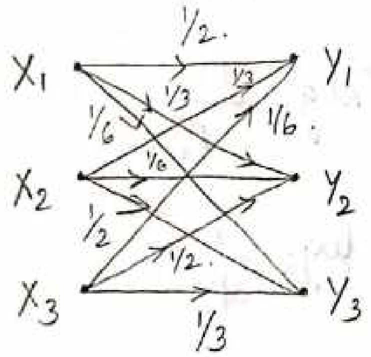
(6) Cascaded channel

→ Symmetric / Uniform channel

A channel is said to be symmetric if the 2nd and subsequent rows of the channel matrix contain the same elements as that on the 1st row but in a different order.

Ex : $P(Y/X) = \begin{bmatrix} P_1 & P_2 & P_3 & P_4 \\ P_2 & P_1 & P_4 & P_3 \\ P_4 & P_3 & P_1 & P_2 \end{bmatrix}$

$$P(Y/X) = \begin{bmatrix} 1/2 & 1/3 & 1/6 \\ 1/3 & 1/6 & 1/2 \\ 1/6 & 1/2 & 1/3 \end{bmatrix}$$



→ Channel capacity of Symmetric / uniform channel

PT $C = \log_2 S - h$

WKT the channel matrix is given by,

$$P(Y/X) = \begin{array}{c} \\ X_1 \\ X_2 \\ \\ \vdots \\ X_n \end{array} \begin{array}{c} \begin{array}{cccc} Y_1 & Y_2 & \cdots & Y_m \end{array} \\ \begin{bmatrix} P_1 & P_2 & P_3 & \cdots & P_m \\ P_2 & P_3 & P_4 & \cdots P_m & P_1 \\ P_3 & P_4 & \cdots & & P_2 \\ \vdots & & & & \\ P_m & P_1 & \cdots & & P_{m-1} \end{bmatrix} \end{array}$$

where $P_1, P_2 \ldots P_m$ are the conditional probs $P(Y_j/X_i)$.

$$\sum_{j=1}^{m} P(y_j/x_i) = 1$$

From the matrix we have,

$$\sum_{j=1}^{m} P(y_j) = 1 \quad \text{or} \quad \sum_{j=1}^{m} P_j = 1$$

we have,

$$H(Y/X) = \sum_i \sum_j P(x_i, y_j) \log_2 \frac{1}{P(y_j/x_i)}$$

$$H(Y/X) = \sum_i \sum_j P(y_j/x_i) \cdot P(x_i) \log_2 \frac{1}{P(y_j/x_i)}$$

$$= \sum_i \sum_j P_j P(x_i) \log_2 \frac{1}{P(y_j/x_i)}$$

$$= \sum_i \sum_j P_j P(x_i) \log_2 \frac{1}{P_j}$$

$$= \sum_i P(x_i) \sum_j P_j \log_2 \frac{1}{P_j}$$

$$= \sum_j P_j \log_2 \frac{1}{P_j}$$

Let $\sum_j P_j \log_2 \frac{1}{P_j} = h$ (constant)

Since it remains same for all the rows.

WKT $C = \text{Max}(I(X,Y))\, r_s$.

Let $r_s = 1$

$$C = \text{Max}[H(Y) - H(Y/X)]$$

$$= \text{Max}[H(Y)] - h$$

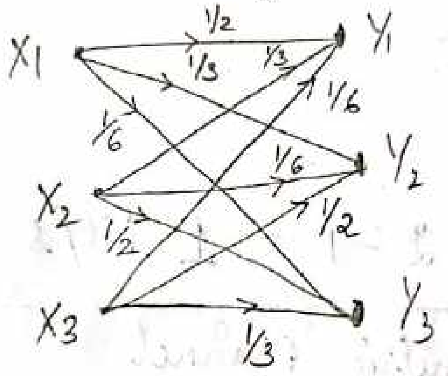
$$\boxed{C = \log_2 m - h}$$

H(Y), which is the entropy of the o/p symbols becomes max iff the received symbol become equiprobable, there are M o/p symbols.

$\therefore$ $\text{Max}[H(Y)] = \log_2 m$

Hence $\boxed{C = \log_2(m) - h}$

→ For the channel matrix shown, find the channel capacity :- Assume $r_s = 1000$ symbols/s.



$$P(Y/X) = \begin{bmatrix} 1/2 & 1/3 & 1/6 \\ 1/3 & 1/6 & 1/2 \\ 1/6 & 1/2 & 1/3 \end{bmatrix}$$

$$h = \sum_j P_j \log_2 \frac{1}{P_j}$$

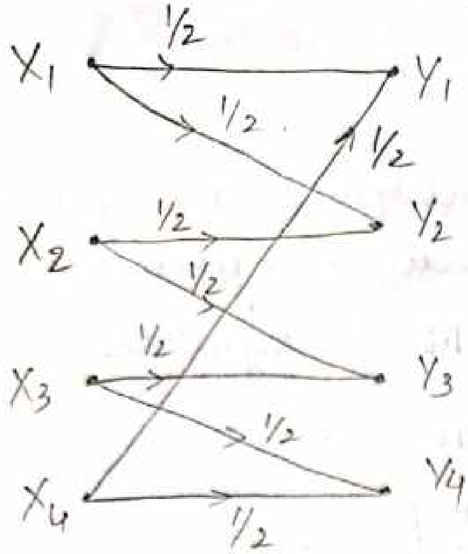
$$= \frac{1}{2} \log_2 \frac{1}{1/2} + \frac{1}{3} \log_2 3 + \frac{1}{6} \log_2 6.$$

$= 1.459$ bps.

$C = \log_2 3 - 1.459 = 1.5849 - 1.459$

$C = 0.1259 \times r_s = 125.9$ bits/s.

→ Determine the capacity for the channel diagram shown below.



$$P(Y/X) = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 1/2 & 0 & 0 & 1/2 \end{bmatrix}$$

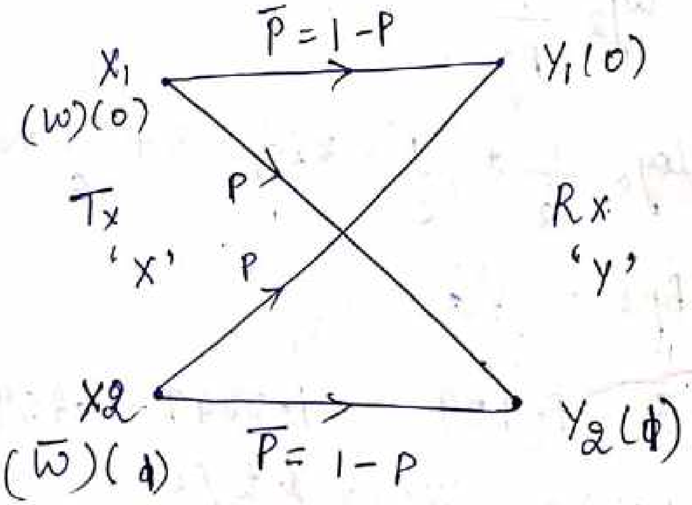
$$h = \sum_j P_j \log_2 \frac{1}{P_j}$$

$$= 2 \times \frac{1}{2} \log_2 2$$

$$h = 1 \text{ bps}$$

$$C = \log_2 4 - h = 2 - 1 = 1 \text{ bit/s.}$$

16/10/17 **Binary Symmetric Channel**



$$C = I(X, Y)\, r_s$$

consider the binary symmetric channel shown above. Let $P(X_1) = w$ & $P(X_2) = \bar{w} = 1 - w$

Let P be the probability of error. ie Prob[18] of receiving 1 when 0 is transmitted or receiving 0 when 1 is transmitted.

From the fig, the channel matrix is given by $P(Y/X) = \begin{bmatrix} \bar{P} & P \\ P & \bar{P} \end{bmatrix}$

$\therefore H(Y/X) = \sum_j P_j \log_2 \frac{1}{P_j} = h$

$$H(Y/X) = \bar{P} \log_2 \frac{1}{\bar{P}} + P \log_2 \frac{1}{P} \longrightarrow (1)$$

To find $H(Y) = \sum_j P(Y_j) \log_2 \frac{1}{P(Y_j)}$

$P(Y) = [\bar{P}\omega + P\bar{\omega} \quad P\omega + \bar{P}\bar{\omega}]$

$P(X, Y) = P(Y/X) \; P(X)$

$= \begin{bmatrix} \bar{P} & P \\ P & \bar{P} \end{bmatrix} [\omega \quad \bar{\omega}]$

$P(X, Y) = \begin{bmatrix} \bar{P}\omega & P\omega \\ P\bar{\omega} & \bar{P}\bar{\omega} \end{bmatrix}$

$P(Y) = [\bar{P}\omega + P\bar{\omega} \quad P\omega + \bar{P}\bar{\omega}]$

$$H(Y) = (\bar{P}\omega + P\bar{\omega}) \log_2 \frac{1}{(\bar{P}\omega + P\bar{\omega})} + (P\omega + \bar{P}\bar{\omega}) \log_2 \frac{1}{(P\omega + \bar{P}\bar{\omega})} \longrightarrow (2)$$

WKT if $r_s = 1$, channel capacity

$C = \log_2(m) - h$

For binary channel $m = 2$, $C = 1 - h$

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$$\therefore \quad C = 1 - \left[\bar{P}\log_2 \frac{1}{\bar{P}} + P\log_2 \frac{1}{P}\right] \rightarrow (3)$$

Let $w = \bar{w} = \frac{1}{2}$

$\therefore$ Eqn (2) changes to

$$H(Y) = \frac{1}{2}\left[(\bar{P}+P)\log_2 \frac{2}{(\bar{P}+P)} + (P+\bar{P})\log_2 \frac{2}{(P+\bar{P})}\right]$$

$$= \frac{1}{2}[1+1] = 1$$

$$I(X,Y) = H(Y) - H(Y/X)$$

$$= 1 - \left[\bar{P}\log_2 \frac{1}{\bar{P}} + P\log_2 \frac{1}{P}\right]$$

Channel capacity C : for $r_s = 1$

$$C = 1 - \left[\bar{P}\log_2 \frac{1}{\bar{P}} + P\log_2 \frac{1}{P}\right]$$

The above Eqn gives channel capacity for BSC which is same as (3)

$\rightarrow$ A BSC has the following noise matrix where $P(X_1) = 2/3$, $P(X_2) = 1/3$. $P(Y/X) = \begin{bmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix}$

Determine $H(X)$, $H(Y)$, $H(X,Y)$, $H(Y/X)$, $H(X/Y)$, $I(X,Y)$, C, η, R_s.

__Soln__ $P(X,Y) = P(Y/X) \cdot P(X)$

$$= \begin{bmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix} \begin{bmatrix} 2/3 & 1/3 \end{bmatrix}$$

$$P(X,Y) = \begin{bmatrix} 1/2 & 1/6 \\ 1/12 & 1/4 \end{bmatrix}$$

$$H(X,Y) = \frac{1}{2}\log_2 2 + \frac{1}{6}\log_2 6 + \frac{1}{4}\log_2 4 + \frac{1}{12}\log_2 12$$

$H(X,Y) = 1.7295$ bps.

$$H(Y) = \sum_j P(Y_j)\log_2 \frac{1}{P(Y_j)}$$

$P(Y) = [7/12 \quad 5/12]$

$$H(Y) = \frac{7}{12}\log_2 \frac{12}{7} + \frac{5}{12}\log_2 \frac{12}{5}.$$

$= 0.9798$ bps.

$$H(X) = \sum_i P(X_i)\log_2 \frac{1}{P(X_i)}$$

$$= \frac{2}{3}\log_2 \frac{3}{2} + \frac{1}{3}\log_2 3$$

$H(X) = 0.918$ bps

$H(Y/X) = ?$

$H(Y/X) = H(X,Y) - H(X).$

$= 1.7295 - 0.918$

$= 0.8115$ bps

$H(X/Y) = H(X,Y) - H(Y)$

$= 1.7295 - 0.9798$

$= 0.7497$ bps.

$I(X,Y) = H(X) + H(Y) - H(X,Y)$

$= 0.918 + 0.9798 - 1.7295$

$= 0.1683$ bps.

$C = 1 - h = 1 - 0.8115.$

$C = 0.1885$

$$\eta_s = \frac{H(X) - H(X/Y)}{C} \times 100\%$$

$$= \frac{0.918 - 0.7497}{0.1885} \times 100\%$$

$$= 89.28\%$$

$R_s = 10.72\%$. ie $R_s = 1 - \eta_s$.

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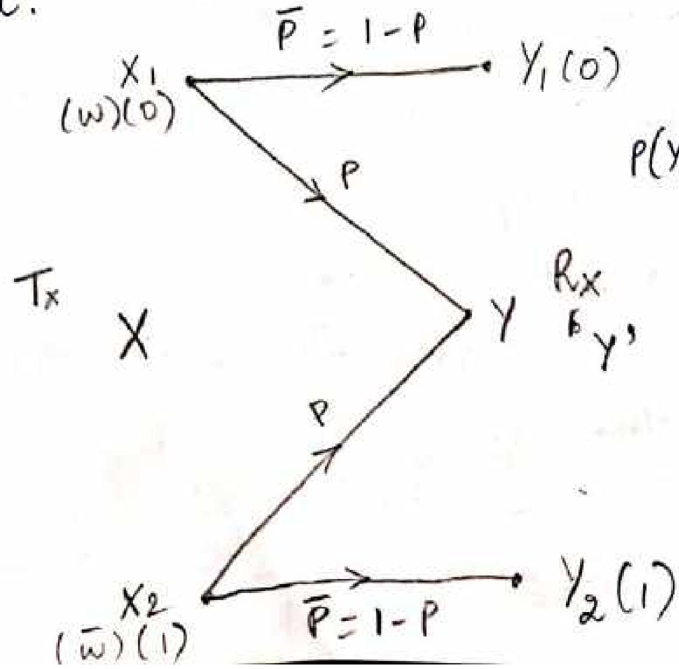
→ Binary Erasure Channel

Binary Erasure channel is one in which whenever an error occurs, the symbol will be received as y & no decision will be made about the info but an immediate request will be made to a reverse channel for retransmission [ARQ system - automatic repeat request] of the transmitted signal till a correct symbol is received at the output. This ensures 100% correct data recovery. Since the error is erased in this channel, It is called as Binary Erasure channel.

The disadvantage is the requirement of a reverse channel.



$$P(Y/X) = \begin{bmatrix} \bar{P} & P & 0 \\ 0 & P & \bar{P} \end{bmatrix}$$

$$I(X,Y) = H(X) - H(X/Y).$$

The channel capacity of BEC is given by.

$$C = Max\,[I(X,Y)] \longrightarrow (1)$$

but $I(X,Y) = H(X) - H(X/Y) \longrightarrow (2)$

$$H(x) = \sum_i P(X_i) \log_2 \frac{1}{P(X_i)}$$

$P(X_1) = \omega$

$P(X_2) = \bar{\omega}$

$$H(X) = \omega \log_2 \frac{1}{\omega} + \bar{\omega} \log_2 \frac{1}{\bar{\omega}}$$

$$P(X,Y) = P(Y/X) \cdot P(X).$$

$$= \begin{bmatrix} \bar{P} & P & 0 \\ 0 & P & \bar{P} \end{bmatrix} [\omega, \bar{\omega}]$$

$$= \begin{bmatrix} \bar{P}\omega & P\omega & 0 \\ 0 & P\bar{\omega} & \bar{P}\bar{\omega} \end{bmatrix}$$

$$P(Y) = [\bar{P}\omega \quad P\omega + P\bar{\omega} \quad \bar{P}\bar{\omega}] = \bar{P}\omega \quad P \quad \bar{P}\bar{\omega}$$

$$P(X/Y) = \frac{P(X,Y)}{P(Y)} = \begin{bmatrix} 1 & \frac{P\omega}{P\omega + P\bar{\omega}} & 0 \\ 0 & \frac{P\bar{\omega}}{P\omega + P\bar{\omega}} & 1 \end{bmatrix} = \begin{bmatrix} 1 & \omega & 0 \\ 0 & \bar{\omega} & 1 \end{bmatrix}$$

$$P(X/Y) = \begin{bmatrix} P(X_1/Y_1) & P(X_1/Y) & P(X_1/Y_2) \\ P(X_2/Y_1) & P(X_2/Y) & P(X_2/Y_2) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{P(X_1,Y_1)}{P(Y_1)} & \frac{P(X_1,Y)}{P(Y)} & \frac{P(X_1,Y_2)}{P(Y_2)} \\ \frac{P(X_2,Y_1)}{P(Y_1)} & \frac{P(X_2,Y)}{P(Y)} & \frac{P(X_2,Y_2)}{P(Y_2)} \end{bmatrix}$$

$$H(X/Y) = \sum_i \sum_j P(X_i, Y_j) \log_2 \frac{1}{P(X_i/Y_j)}$$

$$= \bar{P}\omega \log_2 1 + P\omega \log_2 \frac{1}{\omega} + P\bar{\omega} \log_2 \frac{1}{\bar{\omega}} + \bar{P}\bar{\omega} \log_2 \frac{1}{1}$$

$$= \bar{P}\omega \cdot 0 + P\omega \log_2 \frac{1}{\omega} + P\bar{\omega} \log_2 \frac{1}{\bar{\omega}} + 0$$

$$H(X/Y) = P\omega \log_2 \frac{1}{\omega} + P\bar{\omega} \log_2 \frac{1}{\bar{\omega}}$$

$$= P\left[\omega \log_2 \frac{1}{\omega} + \bar{\omega} \log_2 \frac{1}{\bar{\omega}}\right]$$

$$H(X/Y) = P \cdot H(X)$$

$$I(X, Y) = H(X) - H(X/Y)$$

$$= H(X) - P\,H(X)$$

$$= H(X)\,[1 - P]$$

$$I(X, Y) = \bar{P}\,H(X)$$

Using the above Eq^n in (1)

$$C = \text{Max}\,[\bar{P}\,H(X)]$$

$$C = \bar{P} \cdot H(X)_{max}$$

$$= \bar{P} \log_2 n$$

$$= \bar{P} \cdot \log_2(2)$$

$$\boxed{C = \bar{P}}$$

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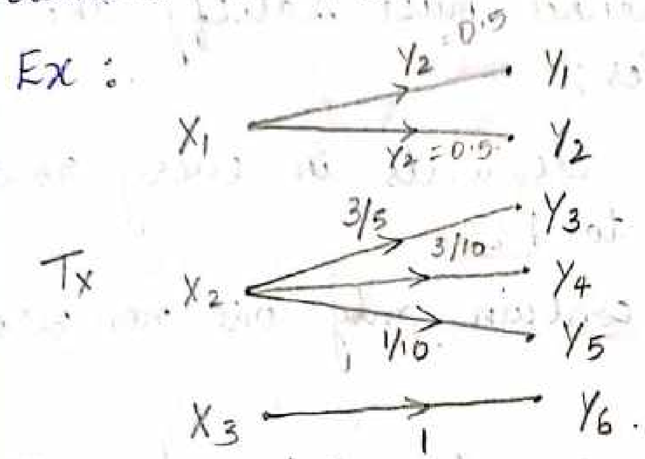
→ Noiseless Channel

A channel ~~matrix~~ represented by a channel matrix with 1 & only 1 non zero element in every column is defⁿ as noiseless channel.

It has 2 properties : (a) The sum of all elements in any row is equal to one.
(b) Each column contains only one non zero element. ie ~~H(Y~~

Ex :



$$P(Y/X) = \text{Rx}\begin{bmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3/5 & 3/10 & 1/10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(i) Noiseless : $H(X/Y) = 0$.

(ii) Deterministic : $H(Y/X) = 0$.

$$C = \text{Max}[I(X,Y)]$$

$$I(X,Y) = H(X) - H(X/Y)$$

$$H(X) = \sum_i P(X_i) \log_2 \frac{1}{P(X_i)}$$

$$H(X) = P(X_1)\log_2 \frac{1}{P(X_1)} + P(X_2)\log_2\frac{1}{P(X_2)} + P(X_3)\log_2\frac{1}{P(X_3)}$$

$$P(X,Y) = P(Y/X) \cdot P(X)$$

$$P(X/Y) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$H(X/Y) = \log_2 1 = 0$

$C = \log_2 3 = 1.58$

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→ **Deterministic Channel:**

A channel represented by a channel matrix with one & only one non zero element in every row is called as Deterministic channel.

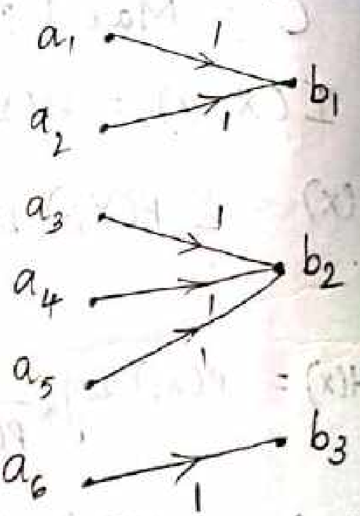
A deterministic channel must satisfy the following properties:

1) The sum of the elements in every row must be equal to 1.

2) Each row must contain only one non zero entity.

Ex: For the given channel matrix, draw the channel diagram & calculate $H(B/A)$. Also find the $I(A, B)$ & the channel capacity.

$$P(B/A) = \begin{array}{c c} & \begin{array}{ccc} b_1 & b_2 & b_3 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{array} & \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$



$H(B/A) = ?$

$$H(B/A) = \sum_i \sum_j P(A_i, B_j) \log_2 \frac{1}{P\left(\frac{B_j}{A_i}\right)}$$

$$P(A, B) = P(B/A) \cdot P(A)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} P(A_1) & P(A_2) & P(A_3) \cdot P(A_4) & P(A_5) & P(A_6) \end{bmatrix}$$

$$= \begin{bmatrix} P(A_1) & 0 & 0 \\ P(A_2) & 0 & 0 \\ 0 & P(A_3) & 0 \\ 0 & P(A_4) & 0 \\ 0 & P(A_5) & 0 \\ 0 & 0 & P(A_6) \end{bmatrix}$$

$$H(B/A) = P(A_1) \log_2 \frac{1}{1} + \ldots$$

$$H(B/A) = 0$$

$$I(A;B) = H(B) - H(B/A)$$
$$= H(B).$$

$$C = \text{Max}\,[I(A,B)]$$
$$= \text{Max}\,[H(B)]$$
$$C = \log_2 3 = 1.58$$

→ <u>CASCADED CHANNELS</u>

Consider 2 discrete channels as shown below

$$X \longrightarrow \boxed{\text{Channel I}} \xrightarrow{\;Y\;} \boxed{\text{Channel II}} \longrightarrow Z$$

For channel I, $I(X,Y) = H(Y) - H(Y/X)$

together for both the channels,

$$I(X,Z) = H(Z) - H(Z/X)$$

then $I(X,Z) < I(X,Y)$

→ Estimation of channel capacity by MUROGA's Method

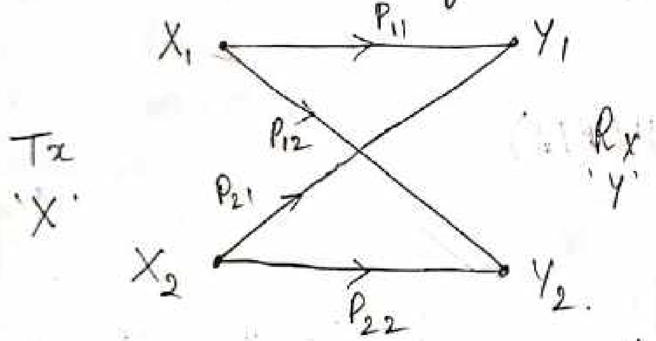
This method can be used to find the channel capacity provided $P(Y/X)$ is a square matrix & Non-singular matrix.

If A^{-1} exists then it is called as non singular matrix.

If $|A| = 0$, then it is singular matrix.

In general, $C = \log_2 \sum_{i=1}^{n} 2^{+Q_i}$

Consider a binary channel as shown:



$$P(Y_j/X_i) = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

Using the above method,

$$C = \log_2 [2^{Q_1} + 2^{Q_2}]$$

Q_1 & Q_2 can be found using the relationship

$$\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} P_{11} \log_2 P_{11} + P_{12} \log_2 P_{12} \\ P_{21} \log_2 P_{21} + P_{22} \log_2 P_{22} \end{bmatrix}$$

$$P_{11} Q_1 + P_{12} Q_2 = P_{11} \log_2 P_{11} + P_{12} \log_2 P_{12} \rightarrow (1)$$

$$P_{21} Q_1 + P_{22} Q_2 = P_{21} \log_2 P_{21} + P_{22} \log_2 P_{22} \rightarrow (2)$$

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Solving ① & ② we get Q_1 & Q_2, which can be used to find the channel capacity

In general, $C = \log_2 [2^{Q_1} + 2^{Q_2}]$.

$$\begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ P_{21} & \cdots & & P_{2n} \\ \vdots & & & \\ P_{n1} & \cdots & & P_{nn} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_n \end{bmatrix} = \begin{bmatrix} P_{11} \log P_{11} + \cdots + P_{1n} \log P_{1n} \\ P_{21} \log P_{21} + \cdots + P_{2n} \log P_{2n} \\ \vdots \\ P_{n1} \log P_{n1} + \cdots + P_{nn} \log_2 P_{nn} \end{bmatrix}$$

$$C = \log_2 [2^{Q_1} + 2^{Q_2} + \cdots + 2^{Q_n}]$$

→ Probability

→ Consider a BSC whose channel matrix,

$P(Y/X) = \begin{bmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix}$ Find channel capacity using Muroga's method

Soln

$$\begin{bmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} 3/4 \log_2 3/4 + 1/4 \log_2 1/4 \\ 1/4 \log_2 1/4 + 3/4 \log_2 3/4 \end{bmatrix}$$

$$\frac{3}{4} Q_1 + \frac{1}{4} Q_2 = -(0.3113 + 0.5) = -0.8113$$

$$\frac{1}{4} Q_1 + \frac{3}{4} Q_2 = -(0.5 + 0.3113) = -0.8113$$

$$\frac{9}{4} Q_1 + \cancel{\frac{3}{4} Q_2} = -2.4339$$

$$-\frac{1}{4} Q_1 + \cancel{\frac{3}{4} Q_2} = +0.8113$$

$$2Q_1 = -1.6226$$

$$Q_1 = -0.8113 = Q_2$$

$$C = \log_2 [2^{-0.8113} + 2^{-0.8113}]$$

$$= 0.188$$

→ Continuous Channels

Generally, the info that is normally trans-mitted is a continuous speech or picture signal rather than discrete. Hence the info can be thought of in a continuous sample space. The sample pts will form a continuous, in contrast to discrete.

The continuous channel can be defⁿ as the one with the i/p belonging to a continuous sample space & the output is also a sample pt belonging to either the same or a different sample space.

A zero memory continuous channel is defⁿ as the one in which the channel o/p statistically depends on the corresponding channels without memory.

Entropy of Continuous Signals [Differential entropy]

We defⁿ the entropy of a continuous channel in terms of statistical averages of source of prob distⁿ.

WKT, the entropy for discrete source is given by $H(S) = \sum_i P_i \log_2 \frac{1}{P_i}$

Consider a cont. random variable X_i with prob density function $f(x)$, then the entropy is given by:

$$H(X) = \int_{-\infty}^{\infty} f(x) \log_2 \frac{1}{f(x)} \, dx$$

Mutual Info & Capacity of Cont. channels

We have, $H(X,Y) = \int\int_{-\infty}^{\infty} f(x,y) \log_2 \frac{1}{f(x,y)} \, dx\,dy$

$$H(X/Y) = \int\int_{-\infty}^{\infty} f(x,y) \log_2 \frac{1}{f(x/y)} \, dx\,dy$$

The Mutual Info $I(X,Y)$ is given by:

$$I(X,Y) = H(X) - H(X/Y)$$

$$= \int_{-\infty}^{\infty} f(x) \log_2 \frac{1}{f(x)} \, dx - \int\int_{-\infty}^{\infty} f(x,y) \log_2 \frac{1}{f(x/y)} \, dx\,dy$$

$$= \int_{-\infty}^{\infty}\int f(x)\, f(x/y) \log_2 \frac{1}{f(x)} \, dx\,dy -$$

$$\int\int_{-\infty}^{\infty} f(x,y) \log_2 \frac{1}{f(x/y)} \, dx\,dy$$

$$= \int\int_{-\infty}^{\infty} f(x,y) \log_2 \frac{1}{f(x)\, f(x/y)} \, dx\,dy$$

The channel capacity $C = \text{Max}\,[I(x,y)]$

$$\int_{-\infty}^{\infty} f(x/y) = 1$$

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Module 4

Error Controlled Coding

Introduction

There are different kinds of coding techniques which help us to achieve lower value of avg length. Thereby increasing the coding efficiency. The disadvantage with this type of coding is that there are variable length codes.

Another disadvantage is that the o/p data rates measured over short time periods will fluctuate.

When fixed length codes are used, a single error will affect only that block which can be easily detected and corrected. To detect and correct the errors, error control coding techniques are used that relay on systematic addition of redundant symbols.

→ Coding & Types of Codes.

The key system parameters available in designing cost effective & reliable digital communication System are signal power & channel bandwidth. These 2 along with PSD (Power Spectral density) of noise (η), Determine the bit signal energy to noise power ratio. $\left(\frac{E_b}{N_0}\right)$. This ratio determines the bit error rate.

Error control coding is the calculated use of redundancy.
The Error control coding improves the data

quality to a great extent and also reduction in $\frac{E_b}{N_0}$ for a fixed bit error rate.

The disadvantage of error control coding are

(a) Increased bandwidth

(b) System becomes more complex due to implementation of decoding operⁿˢ in the receiver

The channel encoder at the transmitter systematically adds bits to the transmitted message, these additional bits carry no info but make it possible for the channel decoder to detect & correct errors in the information bits. This reduces the overall probability of error (P_e). These additional bits are called redundant bits.

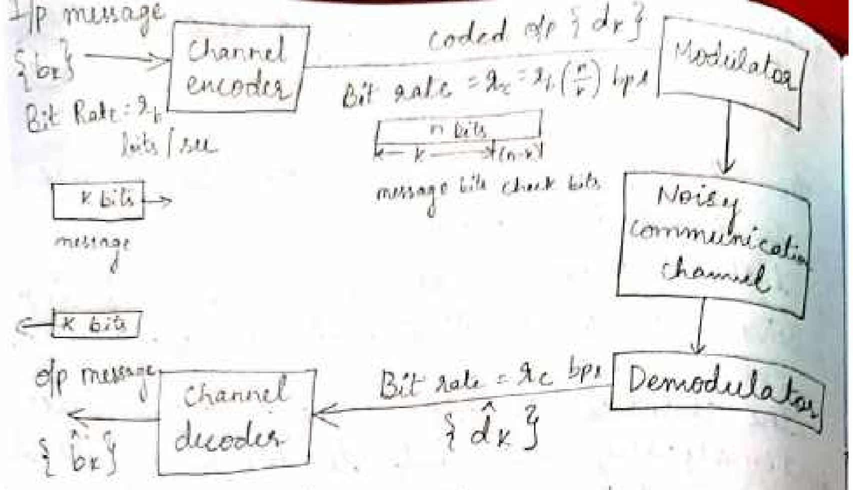
Types of Codes:

1. Block codes — no memory / storage
2. Convolution codes — memory is used, storage is necessary (use of FF ↓ flip flops.)

Another way of classifying codes is linear or non linear. A linear code differs from non linear code by the property that any 2 code words added using modulo 2 arithematic produces a 3rd code word.

→ Example of Error Control Coding

The source generates a message block $\{b_k\}$ at a rate of r_b bits/sec & feeds it to the channel encoder.



Bill rate $= r_b$ bits/sec

The channel encoder then adds $(n-k)$ no. of check bits or redundant bits to these k bit message to form 'n' bit code word.

These $(n-k)$ check bits donot carry any info but helps the channel decoder to detect & correct errors.

The bit rate of the coded o/p block will be

$$r_c = r_b\left(\frac{n}{k}\right) \text{ bits/sec.}$$

The channel decoder decodes $\{\hat{d}_k\}$ to get back the info block $\{\hat{b}_k\}$ at the receiver.

The probability of error $P_e = P\{b_k \neq \hat{b}_k\}$

→ Methods of Controlling Errors

There are 2 methods available :

1. **Forward acting error correction method**

The method of controlling errors at the receiver through attempts to correct noise induced errors is called Forward acting error correction

2. Error Detection method :

In this method, the decoder examines the demodulator o/p, accepts the received sequence if it matches with a valid message sequence. If not the decoder discards the received sequence and notifies the transmitter (through a reverse channel) regarding the error & requests for retransmission of the message till the correct seq^n is received ie decoder attempts to detect the error but doesnot correct them.

The disadvantage of this method is the requirement of a reverse channel which will slow down the effective rate of data transmission.

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→ MATRIX Description of Linear Block codes.

The msg block of 'K' bits is represented as a row vector or K-tuple (tuple) msg vector.

$[D] = [d_1\ d_2\ \dots\ d_k]$

Where $d_1, d_2, \dots, d_k$ → are either 0's or 1's.

Thus there are 2^k distinct msg vectors.

The channel encoder systematically adds $(n-k)$ no of check bits to form (n, k) linear block code. Then 2^k code vectors are represented by

$C = [c_1;\ c_2\ \dots\ c_n]$

From the above Eq^n, it is clear that there are only 'k' msg vectors which are valid out of 'n' possibilities ie the remaining $(2^n - 2^k)$ code vectors are invalid code vectors, which form the error vectors.

The ratio $\frac{k}{n}$ is called the rate efficiency of n, k linear block code.

In a <u>systematic</u> linear block code, the msg bits appear either at the beginning or end of the code word.

If the ~~code word~~ message is at the beginning of the code,

$c_i = d_i$, where $i = 1, 2, \ldots k$.

The remaining $(n-k)$ bits are check bits.

$$[C] = [\underbrace{c_1, c_2, c_3 \ldots c_k}_{k \text{ message bits}}, \underbrace{c_{k+1}, c_{k+2}, \ldots c_n}_{(n-k) \text{ check bits}}]$$

These $(n-k)$ no. of check bits are derived from k message bits using predetermined rule given below.

$$c_{k+1} = P_{11} d_1 + P_{21} d_2 + \ldots + P_{k1} d_k$$

$$c_{k+2} = P_{21} d_1 + P_{22} d_2 + \ldots + P_{k2} d_k$$

$$\vdots$$

$$c_n = P_{1,(n-k)} d_1 + \ldots + P_{k,(n-k)} d_k$$

Expressing the above Eqns in the form of matrix we have;

$$[c_1, c_2, \ldots c_k, c_{k+1}, \ldots c_n] = [d_1 d_2 \ldots d_k] \left[\begin{array}{c|c} \overbrace{\begin{matrix} 1\,0 \ldots 0 \\ 0\,1\,0 \ldots 0 \\ \vdots \\ 0 \ldots 1 \end{matrix}}^{k \text{ terms}} & \overbrace{\begin{matrix} P_{11} & P_{12} & \ldots & P_{1,n-k} \\ P_{21} & P_{22} & \ldots & P_{2,n-k} \\ \vdots \\ P_{k1} & P_{k2} & \ldots & \end{matrix}}^{(n-k) \text{ terms}} \end{array}\right]$$

$[C] = [D][G]$

$$\boxed{[G] = [I_K \vdots P]_{K \times n}}$$

Matrix G ie $[G] \longrightarrow$ generator matrix.

$[P] \longrightarrow$ arbitrary matrix called as parity matrix of order $K \times (n-K)$.

Also $[G]$ can be written as $[P \vdots I_K]$

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4 \times 4}.$$

$\longrightarrow$ For a systematic $[6,3]$ linear block code; the parity matrix $[P] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. Find all the possible code vectors.

__Soln__ $[C] = [D][G]$ $\quad [6, 3] = [n, K]$

$[G] = [I_K \vdots P]_{K \times n}$

$$[G] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]_{K \times n}$$

$0+0=0$

$0+1=1$

$1+0=1$

$1+1=0$

$0-1=1$

$$[C] = [d_1 \; d_2 \; d_3] \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

$$= [d_1, d_2, d_3, d_1+d_3, d_2+d_3, d_1+d_2]$$

Code Name	Message Vector	Code Vector
C_1	000	000000
C_2	001	001110
C_3	010	010011
C_4	011	011101

C_5	100	100101
C_6	101	101011
C_7	110	110110
C_8	111	111000

$$[C] = [0\,0\,1]\begin{bmatrix} 1&0&0&1&0&1 \\ 0&1&0&0&1&1 \\ 0&0&1&1&1&0 \end{bmatrix} = [0\,0\,1\,1\,1\,0]$$

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*	Addition	Subtraction
0+0 =	0	0
0+1 =	1	1
1+0 =	1	1
1+1 =	0	0

$$\boxed{\begin{aligned} C &= A \ominus B \\ &= A \oplus B \end{aligned}}$$

→ **Parity Check Matrix**

WKT $[G] = [I_K \,\vdots\, P]_{K \times n}$

$$[G] = \left[\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & P_{11} & P_{12} & \cdots & P_{1,(n-k)} \\ 0 & 1 & \cdots & 0 & P_{21} & P_{22} & \cdots & P_{2,(n-k)} \\ \vdots & & & & \vdots & & & \\ 0 & \cdots & \cdots & 1 & P_{K1} & P_{K2} & \cdots & P_{K,(n-k)} \end{array}\right]$$

The parity check matrix H is given by:

$$H = [P^T \,\vdots\, I_{n-K}]_{(n-k)\times n}$$

$$= \left[\begin{array}{cccc|cccc} P_{11} & P_{21} & \cdots & P_{K1} & 1 & 0 & \cdots & 0 \\ P_{12} & P_{22} & & P_{K2} & 0 & 1 & \cdots & 0 \\ \vdots & & & \vdots & \vdots & & & \\ P_{1,(n-k)} & \cdots & \cdots & P_{K,(n-k)} & 0 & \cdots & \cdots & 1 \end{array}\right]$$

→ PT, $C \cdot H^T = 0$

Soln $LHS = C \cdot H^T$

$= DGH^T \longrightarrow (1)$

Consider GH^T, $g_i = [0\,0\,0 \cdots \underset{i^{th}\text{ element}}{1} \cdots 0 \,|\, P_{i1}\, P_{i2} \cdots \underset{(k+j)}{P_{ij}} \cdots P_{i(n-k)}]$

$g_i = [0\,0\,0 \cdots 1 \cdots 0 \,|\, P_{i1}\, P_{i2} \cdots P_{ij} \cdots P_{i(n-k)}]$

i^{th} element $\quad (k+j)^{th}$ element

$H_j = [P_{1j}\, P_{2j} \cdots P_{ij} \cdots P_{kj} \,|\, 0\,0 \cdots 1 \cdots 0]$

$P_{1j}\, P_{2j} \cdots P_{ij} \cdots$

i^{th} element $\quad (k+j)^{th}$ element

$$g_i H_j^T = [0\,0 \cdots 1 \cdots 0 \,|\, P_{i1}\, P_{i2} \cdots P_{ij} \cdots P_{i(n-k)}] \begin{bmatrix} P_{1j} \\ P_{2j} \\ \vdots \\ P_{ij} \\ \vdots \\ P_{kj} \\ \hline 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

i^{th} element $\quad (k+j)^{th}$

$= P_{ij} \oplus P_{ij} \quad \begin{bmatrix} 1+1=0 \\ 0+0=0 \end{bmatrix}$

$= 0$

Hence Proved

$\forall\, i, j,\ g_i H_j^T = 0$

ie $[G][H]^T = 0$

using this in Eqn (1)

we have $LHS = GH^T = 0$.

Hence Proved.

→ Encoding circuit for (n, k) linear block code.

WKT $C = DG$

$$C = [d_1\ d_2\ \dots\ d_k] \begin{array}{cc} \overbrace{\qquad\qquad}^{k \text{ elements}} & \overbrace{\qquad\qquad\qquad}^{(n-k) \text{ elements}} \end{array}$$
$$\left[\begin{array}{cccc|cccc} 1 & 0 & \dots & 0 & P_{11} & P_{12} & \dots & P_{1(n-k)} \\ 0 & 1 & \dots & 0 & P_{21} & P_{22} & \dots & P_{2(n-k)} \\ \vdots & & & & \vdots & & & \\ 0 & 0 & \dots & 1 & P_{k1} & \dots & \dots & P_{k(n-k)} \end{array}\right]$$

$C_1 = d_1$

$C_2 = d_2$

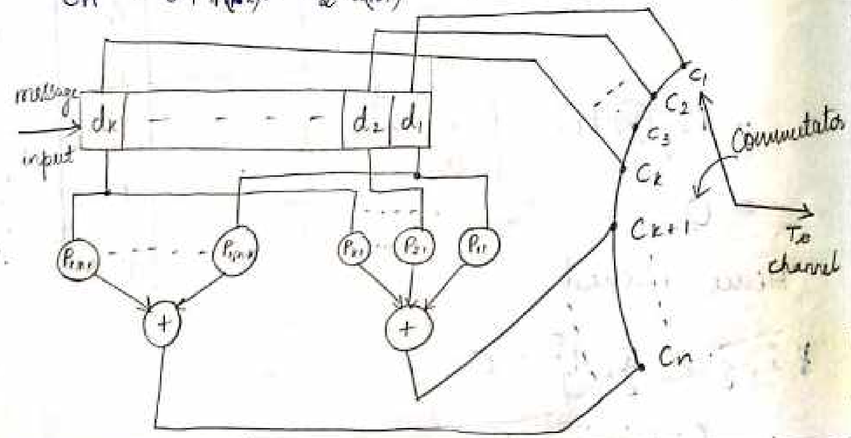
$\vdots$

$C_k = d_k$

$C_{k+1} = d_1 P_{11} + d_2 P_{21} + \dots + d_k P_{k1}$

$\vdots$

$C_n = d_1 P_{1(n-k)} + d_2 P_{2(n-k)} + \dots\ d_k P_{k(n-k)}$



★ $G = [I_k \,\vdots\, P]_{k \times n}$

$H = [P^T \,\vdots\, I_{n-k}]_{(n-k) \times n}$

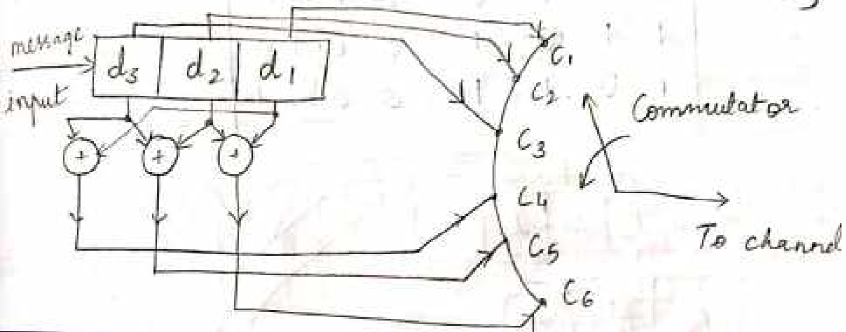
→ For (6, 3) linear block code, $P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Draw the corresponding encoding circuit, $(n, K) = (6, 3)$

Soln $C = DG$, $G = [I_k \mid P]_{k \times n}$; $(n, k) = (6, 3)$

$$= [d_1 \; d_2 \; d_3] \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

$$[C] = [d_1, d_2, d_3, (d_1 + d_3), (d_2 + d_3), (d_1 + d_2)]$$



★ $V = UG$

$U = [v_0 \; v_1 \; \ldots]$

$V = [v_0 \; v_1 \; \ldots]$

→ Consider a systematic [8, 4] LBC whose parity check Eqns are $V_4 = u_1 + u_2 + u_3$

$V_5 = u_0 + u_1 + u_2$; $V_6 = u_0 + u_1 + u_3$,

$V_7 = u_0 + u_2 + u_3$.

① Write the generator & parity check matrices

② Draw the encoder ckt diagram.

Soln $V = UG$

$(n, k) = (8, 4)$

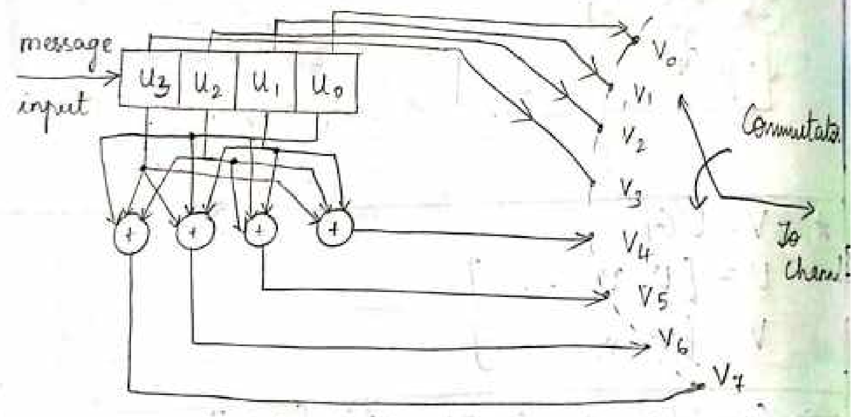
$V_0 = u_0$; $V_2 = u_2$

$V_1 = u_1$; $V_3 = u_3$

$$V = [u_0\ u_1\ u_2\ u_3] \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{array}\right]$$

$$G = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{array}\right] \qquad G = [I_k \mid P]_{k \times n}$$

$$H = \left[\begin{array}{cccc|cccc} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \qquad H = [P^T \mid I_{(n-k)}]_{(n-k)\times n}$$



→ For (7,4) LBC which is in its systematic form, given
$$G = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$
Draw the encoding diagram for corrected code.

Soln $G = [I_k \mid P]_{k \times n}$

$= [I_4 \mid P]_{4 \times 7}$

$$G = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

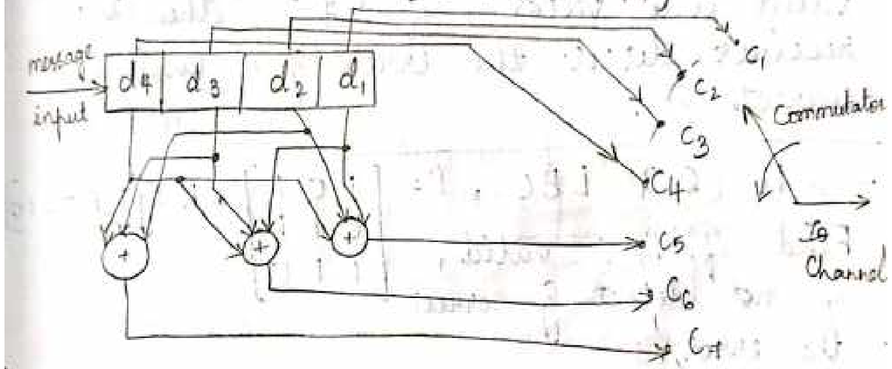
$r_1 \rightarrow r_1 + r_2$

$r_3 \rightarrow r_1 + r_2 + r_3$

$$\Rightarrow G' = \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right]$$

$C = DG$

$$= [d_1\ d_2\ d_3\ d_4] \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right]$$



→ Syndrome Calculation, Error Detection & Correction

Suppose $C = [c_1\ c_2\ \ldots\ c_n]$ be a valid code vector transmitted over a noisy communication channel belonging to (n, k) linear block code.

Let $R = [r_1\ r_2\ \ldots\ r_n]$ be the received vector.

Due to noise in the channel, the received vector may be different from the valid code vector.

The error vector / pattern is def^n as :-

$R = C + E$

$E = R - C = R + C$

$1+1=0$ $\Rightarrow$ $R+C = R-C$

$0+0=0$

$1-1=0$

$0-0=0$

In order to find E & C, the receiver does the decoding operation by determining (n-k) vector "S" defⁿ by :-

$$S = RH^T$$
$$= (C+E)H^T$$
$$= CH^T + EH^T$$
$$\boxed{S = EH^T}$$

$R = C + E$ (C = code, E = error)

$[CH^T = 0]$

If $S = 0$, then the received vector R is a valid code vector, if $S \neq 0$, then the receiver detects the error & if possible, corrects it.

→ For (6,3) LBC, $P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, $R = [1\ 1\ 0\ 0\ 1\ 0]$

Find if R is valid, if not detect & correct the error.

<u>Soln</u> $S = RH^T$

$$H = \left[P^T \,\vdots\, I_{(n-k)}\right]_{(n-k)\times n}$$

$$H = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$S = [1\ 1\ 0\ 0\ 1\ 0] \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [1\ 0\ 0]$$

$S \neq 0$, but the value of $S = [1\ 0\ 0]$ is the 4th row of H^T. ie the 4th bit

counting from left in the received vector
is in error.

$$R = [1\,1\,0\,0\,1\,0] \; ; \; E = [0\,0\,0\,1\,0\,0]$$

$C = R + E$

$$C = [1\,1\,0\,1\,1\,0]$$

→ For (7, 4) LBC, $R = [0\,1\,1\,0\,1\,0\,0]$, the parity matrix, $P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ Detect the error, correct if possible.

<u>Soln</u> $S = RH^T$

$$H = [P^T \,\vdots\, I_{(n-k)}]_{(n-k)\times n}$$

$$H = \left[\begin{array}{cccc|ccc} 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$S = [0\,1\,1\,0\,1\,0\,0] \left[\begin{array}{cccc|ccc} 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]^T$$

$$S = [0\,1\,1\,0\,1\,0\,0] \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [0\,1\,0]$$

$S \neq 0$, but the value of $S = [0\,1\,0]$ is the 6^{th} row of H^T, ie the 6^{th} bit counting from the ~~left~~ in the received matrix/vector is in error.

$$\Rightarrow C = [0\,1\,1\,0\,1\,1\,0]$$

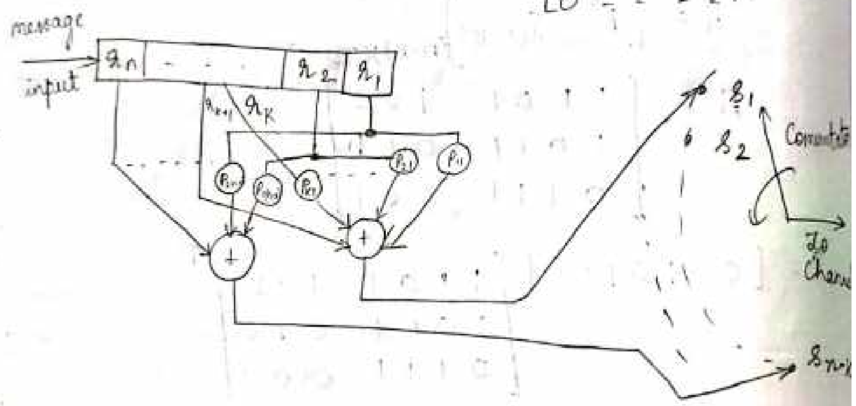
$$R = [0\,1\,1\,0\,1\,0\,0] \; ; \; E = [0\,0\,0\,0\,0\,1\,0]$$

→ Syndrome Calculation Circuit

Let received vector $R = [r_1\ r_2\ \cdots\ r_n]$ then the syndrome $S = [S_1\ S_2\ \cdots\ S_{(n-k)}]$

$S = RH^T$

$$[S_1\ S_2\ \cdots\ S_{n-k}] = [r_1\ r_2\ \cdots\ r_n]\begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1(n-k)} \\ P_{21} & P_{22} & \cdots & P_{2(n-k)} \\ \vdots & & & \vdots \\ P_{k1} & \cdots & \cdots & P_{k(n-k)} \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 1 \end{bmatrix}$$



→ Construct a Syndrome Calculation ckt for (6,3) LBC, given $P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

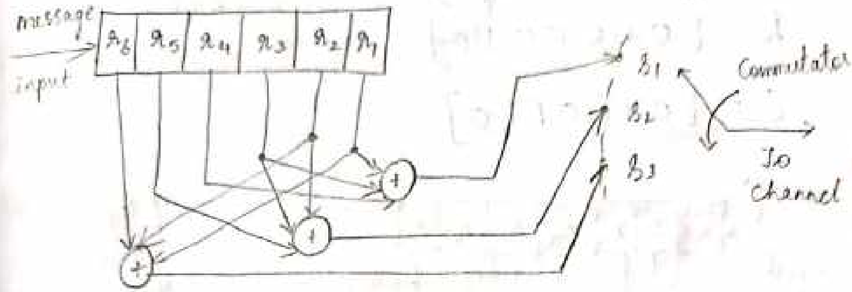
Soln $H = [P^T \vdots I_{(n-k)}]_{(n-k)\times n}$

$$H = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right].$$

$$S = RH^T$$

$$= \begin{bmatrix} r_1 & r_2 & r_3 & r_4 & r_5 & r_6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} r_1 + r_3 + r_4, & r_2 + r_3 + r_5, & r_1 + r_2 + r_6 \end{bmatrix}$$



→ Consider a LBC (7,4), the G matrix =

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Compute the syndrome, if the received vector is $[0\ 1\ 1\ 0\ 1\ 0\ 0]$

Correct the errors if any, Also draw the syndrome calculation circuit.

<u>Soln</u> $G = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$; $P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$$H = [P^T \,\vdots\, I_{(n-k)}]_{(n-k) \times n}$$

$$H = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$S = RH^T = [0110100] \cdot \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [010]$$

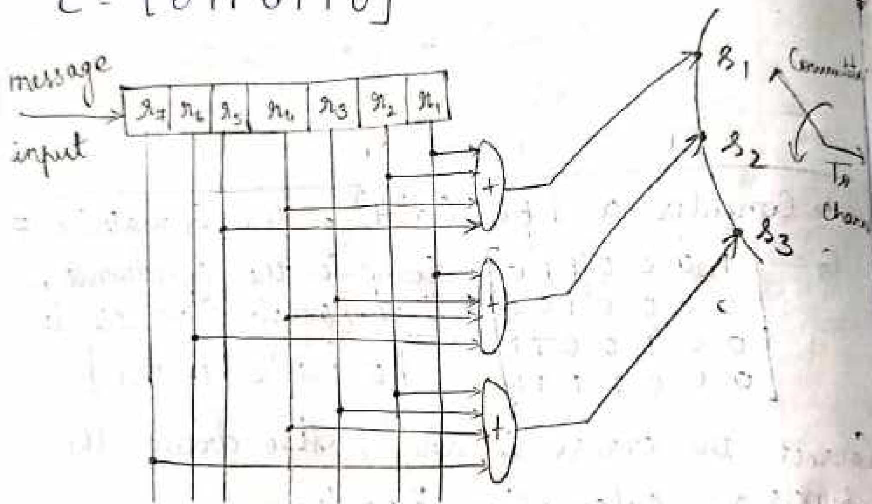
$S = [0\ 1\ 0]$

$S \neq 0$, but the value of $S = [0\ 1\ 0]$ is the 6^{th} row of H^T, ie the 6th bit counting from the left in the received vector is in error.

$R = [0\ 1\ 1\ 0\ 1\ 0\ 0]$ $\quad [C = R + E]$

$E = [0\ 0\ 0\ 0\ 0\ 1\ 0]$

$C = [0\ 1\ 1\ 0\ 1\ 1\ 0]$



→ Definitions :

① **Hamming weight** : Hamming weight of a code vector 'C' is def^n as the no of non zero compo in C. Ex: $C_1 = [0\ 0\ 1\ 0\ 1\ 1\ 0]$

Hamming weight = 3

② **Hamming distance** : It is def^n as the diff^n b/t C_1 & C_2 ie 2 code words.

Ex: $C_1 = [0\ 0\ 1\ 0\ 1\ 1\ 0]$, $C_2 = [1\ 0\ 0\ 1\ 1\ 0\ 0]$

0010110
1001100

Hamming distance = 4

③ Minimu[m]

It is [...] b/t any 2 [...]
Ex: For [...] the Ham[ming] [...]

Message vector
000
001
010
011
100
101
110
111

The m[...]

→ Error [...] of [...]

Theorem

The [...] weight [...]
Consi[der] [...] $j = 1$, [...] dista[nce] [...]
H_w([...]
H_w [...]
Ther[e] [...]

③ Minimum distance [d_{min}]

It is def^n as the smallest hamming dist^n b/t any 2 code vectors in a code.

Ex: For the given message & code vectors, Find the Hamming weight & also minimum distance.

Message vectors	Code vector	Hamming weight	d_{min}
000	000000	0	
001	001110	3	
010	010011	3	
011	011101	4	
100	100101	3	3
101	101011	4	
110	110110	4	
111	111000	3	

The min dist^n (d_{min}) = Hamming weight of a non zero code vector

→ **Error Detection & Error Correction capabilities of Linear block code**

Theorem 5.1

The min dist^n of a LBC. = min Hamming weight of a non zero code vector.

Consider C_i & C_j, where $i = 1, 2, 3 \ldots$ $j = 1, 2, 3 \ldots$. Let $d(C_i, C_j)$ = Hamming distance b/t C_i & C_j.

$H_w(C_i) \rightarrow$ Hamming weight of C_i.

$H_w(C_j) \rightarrow$ Hamming weight of C_j

Then dist^n of $(C_i, C_j) = d(C_i, C_j) = H_w(C_i + C_j)$

Given a Block Code C, one can compute the Hamming distⁿ b/t any 2 distinct code vectors

By definition,

$$d_{min} = \min \{ d(c_i, c_j), c_i \& c_j \in C, c_i \neq c_j \}$$

$$d(c_i, c_j) = H_w(c_i + c_j) = H_w(c_k)$$

ie $d_{min} = \min \{ H_w(c_k) ; c_k \in C \}$

$$\boxed{d_{min} \triangleq H_{w\,min}}$$

Theorem 5.2 (a)

Let 'C' be a (n, k) LBC with parity check matrix H. For each code vector of having weight l, there exists 'l' columns of 'H' such that the vector sum of these 'l' columns = the zero vector

Proof: Consider the parity check matrix 'H'

$$H = [h_1\ h_2\ \dots\ h_n]$$

$h_i \rightarrow i^{th}$ column of H

Let $C = [c_1\ c_2\ \dots\ c_n]$ of having weight 'l'

Then C has 'l' non zero components. ie

$$c_{i1} = c_{i2} = \dots = c_{il} = 1$$

WKT $0 = CH^T$

$$= [c_1\ c_2\ \dots\ c_n] \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{bmatrix}$$

$$0 = c_1 h_1 + c_2 h_2 + \dots + c_n h_n$$

$$0 = c_{i1} h_{i1} + c_{i2} h_{i2} + \dots + c_{il} h_{il}$$

$$0 = h_{i1} + h_{i2} + \dots + h_{il}$$

ie 'l' columns of H add up to 0 vector.

Theorem 5·2 (b) [Converse of Theorem 5·2(a)]

7/11/17 Let C be (n, K) linear block code with parity check matrix H. If there exists 'l' columns of H whose vector sum is 0 vector, then there exists a code vector of hamming weight 'l' in C.

Proof: Let $[h_{i1}, h_{i2} \dots h_{il}]$ be 'l' columns of 'h' such that $h_{i1} + h_{i2} + \dots + h_{il} = 0$

Let us form a binary 'n' tuple whose non zero components are $X_{i1}, X_{i2}, \dots X_{il}$.

The hamming weight of X is 'l'. ~~Con~~

Consider $XH^T = x_1 h_1 + X_2 h_2 + \dots + X_n h_n$

$$XH^T = X_{i1} h_{i1} + X_{i2} h_{i2} + \dots + X_{il} h_{il}$$

But $X_{i1} = X_{i2} = \dots = X_{il} = 1$

$$= h_{i1} + h_{i2} + \dots + h_{il}$$

$$XH^T = 0$$

But $CH^T = 0$

Comparing the above 2 equations, It is clear that X is a code vector of hamming weight 'l' in C.

→ Corollary 5·2 (c)

Let 'C' be a LBC with parity check matrix 'H'. If no $(d-1)$ or ~~are~~ fewer columns of H, add to zero, the code has min hamming weight of atleast d.

01.11.17

→ Generator matrix of a LBC is given by

$$G = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}_{3\times 7}$$

Find
1. Find all possible valid code vectors
2. Find parity check matrix.
3. Find min distⁿ
4. Draw the encoding ckt & syndrome calⁿ ckt.

Soln (1) $C = DG$

$n = 7, \; k = 3$

$$G = [I_k \,\vdots\, P]_{k\times n} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

$D = [d_1 \; d_2 \; d_3]$

Code Name	Message vector	Code Vector	Hamming weight
C_1	000	0000000	0
C_2	001	0011101	4
C_3	010	0100111	4
C_4	011	0111010	4
C_5	100	1001110	4
C_6	101	1010011	4
C_7	110	1101001	4
C_8	111	1110100	4

$$\therefore C = [d_1 \; d_2 \; d_3] \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

$$= [d_1, d_2, d_3, \; d_1+d_3, \; d_1+d_2+d_3, \; d_1+d_2, \; d_2+d_3]$$

(3) $d_{min} = 4$

(3) Parity check matrix

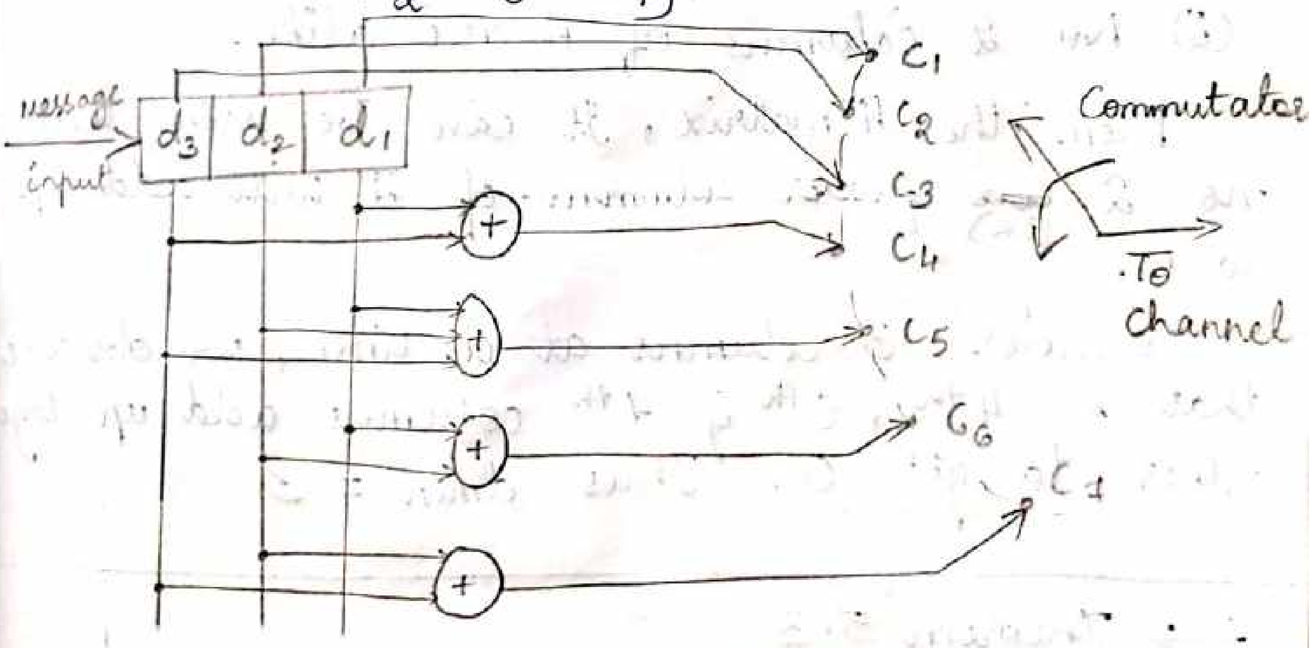
$$H = [P^T \mid I_{(n-k)}]_{(n-k)\times n}$$

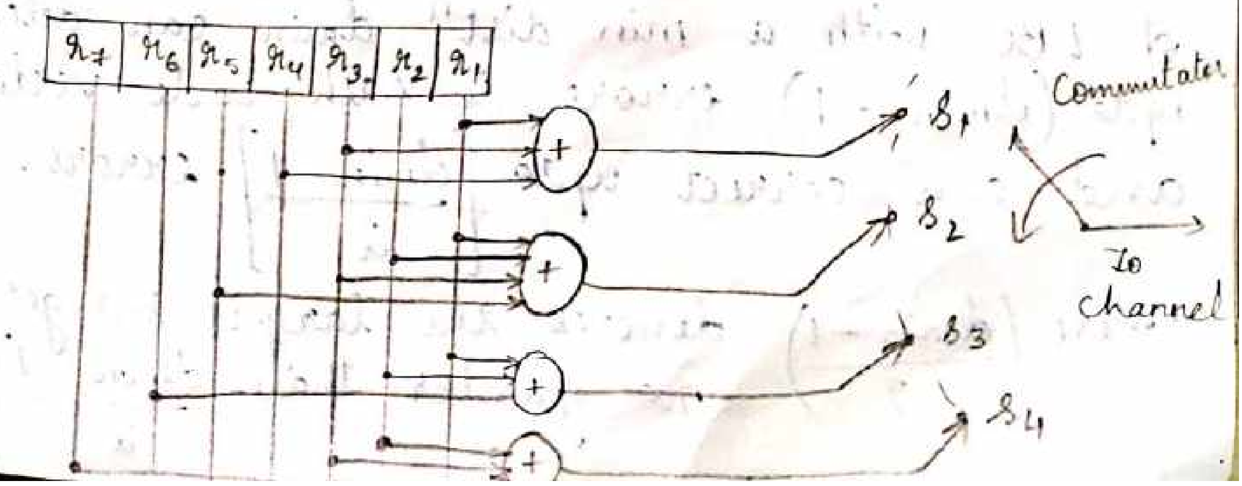
$$H = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(4) $S = RH^T$

$$S = [r_1 \; r_2 \; r_3 \; r_4 \; r_5 \; r_6 \; r_7] \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$S = [r_1 + r_3 + r_4,\; r_1 + r_2 + r_3 + r_5,\; r_1 + r_2 + r_6,\; r_2 + r_3 + r_7]$$





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→ Corollary 5.2(d).

Let C be a LBC with parity check matrix 'H'. The min Hamming weight (= min distⁿ) of C is equal to the smallest no. of columns of H that add upto '0'.

→ Verify Corollary 5.2 (c) & (d).

$$H = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Soln We observe from the H matrix:

(a) All columns of H are non zero columns.

(b) No 2 columns of H are alike.

From the H matrix, It can be seen that no 2 ~~are~~ (OR) fewer columns of H will add up to 0.

Consider 3 columns at a time, we observe that is 4th, 6th & 7th columns add up toge-ther to get 0. Thus $d_{min} = 3$

→ Theorem 5.3

A LBC with a min distⁿ d_{min} can detect upto $(d_{min} - 1)$ errors in each code vector and can correct upto $\left[\frac{d_{min}-1}{2}\right]$ errors.

where $\left(\frac{d_{min}-1}{2}\right)$ denotes the largest integer, no greater than $\left(\frac{d_{min}-1}{2}\right)$

→ Single Error Correcting Hamming Code

Consider the parity check matrix H,

$$H = [P^T \vdots I_{(n-k)}]_{(n-k)\times n}$$

$$H^T = \begin{bmatrix} P \\ \text{---} \\ I_{(n-k)} \end{bmatrix}_{n\times(n-k)} \longrightarrow (1)$$

From Eqⁿ (1), It is clear that there are $(n-k)$ no of columns ie each row of H^T has $(n-k)$ no of entries, each of which could be 0 or 1.

Thus we can have $2^{(n-k)}$ no. of distinct rows.

But in H^T, all zeros cannot be used, as it represents the syndrome of no errors.

$$2^{n-k} - 1 \geq n$$

where $n \rightarrow$ no. of rows in H^T.

$$2^{n-k} \geq n+1$$

$$n-k \geq \log_2 (n+1)$$

$$-k \geq \log_2 (n+1) - n$$

$$k \leq n - \log_2 (n+1)$$

The single error correcting (n,k) Hamming Code has the following properties.

1. Code length : $n \leq 2^{n-k} - 1$
2. No of message bits : $k \leq n - \log_2 (n+1)$

3. No. of parity check bits : (n - k)

4. Error correcting capability : $t = \frac{d_{min} - 1}{2}$.

→ Design (n, k) hamming code with min dist^n, $d_{min} = 3$ & message length of 4 bits. Generate all possible code words & check for 1 bit error correction.

Soln WKT : $n \leq 2^{n-k} - 1$. $\qquad 7 \leq 7$

$n \leq 2^{n-4} - 1$

By trial & error, the least integer value of n which satisfies the above inequality is found to be 7. ie it is a (7, 4) hamming code.

WKT $H^T = \begin{bmatrix} P \\ \hline I_{(n-k)} \end{bmatrix}_{n \times (n-k)} = \begin{bmatrix} P \\ \hline I_3 \end{bmatrix}$

$$H^T = \begin{bmatrix} & P & \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{7 \times 3}$$

The parity matrix P has 4 rows & 3 columns which should be suitably chosen.

The requirements for choosing the 'P' matrix are

(a) H^T should not contain a row of zeros as this represents the syndrome of no error.

(b) No 2 rows of 'H^T' must be same ie all 7 rows of H^T must be distinct.

There are totally $2^{n-k} = 2^{7-4} = 2^3 = 8$ combinations of 3 bit nos. 000 cannot be used & also 100, 010, 001 cannot be used as they are already present in identity matrix. So are left with 011, 101, 110, 111.

ds the 4 rows of P.

Thus we have $4! = 24$ ways to arrange these numbers as rows of P. Any of these 24 combis can be used in H^T which can be used for error correction.

Let $H^T = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow H = \left[\begin{array}{cccc|ccc} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$

$G = [I_K \mid P]_{K \times n}$

$G = [I_4 \mid P]_{4 \times 7}$

$G = \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right]_{4 \times 7}$

$C = DG$

$D = [d_1 \ d_2 \ d_3 \ d_4]$

$C = [d_1 \ d_2 \ d_3 \ d_4] \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right] = [d_1, d_2, d_3, d_4, d_2+d_3+d_4, d_1+d_3+d_4, d_1+d_2+d_4]$

Code Name	Message vector	Code vector	Hamming weight
C_1	0000	0000000	0
C_2	0001	0001111	4
C_3	0010	0010110	3
C_4	0011	0011001	3
C_5	0100	0100101	3
C_6	0101	0101010	3
C_7	0110	0110011	4
C_8	0111	0111100	4
C_9	1000	1000011	3
C_{10}	1001	1001100	3
C_{11}	1010	1010101	4
C_{12}	1011	1011010	4
C_{13}	1100	1100110	4
C_{14}	1101	1101001	4
C_{15}	1110	1110000	3
C_{16}	1111	1111111	7

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Error correction : $d_{min} = 3$

The error correction capability $t = \frac{d_{min} - 1}{2}$

$$t = \frac{3-1}{2} = 1$$

ie (7, 4) hamming code is a single error correcting code (SEC).

Let the received vector with the single er... be $R = [1\,1\,1\,1\,0\,0\,1]$

$$S = RH^T$$

$$S = [1\,1\,1\,1\,0\,0\,1] \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [1\,1\,0]$$

Syndrome $S = [1\,1\,0]$ which is present in 3rd row of H^T which means the third bit in the received vector is in error.

$\therefore$ changing the 3rd bit from 1 to 0, ... have $C = [1, 1\,0\,1\,0\,0\,1]$.

→ **Hamming Bound**

For an (n, k) LBC, we have 2^{n-k} syndr... including all zero syndrome. Each syndrome corresponds to a specific error pattern. If 'i' is the no. of error locations in the ... dimensional error pattern 'e'. We find in general, there are ${}^{n}C_{i}$ multiple error pa... It then follows that the total no. of possible error patterns $= \sum_{i=0}^{t} {}^nC_i$.

$t \rightarrow$ max no of error locations in e.

"If an (n, k) LBC is capable of correcting 't' errors, then the total no. of syndromes shall not be less than the total no of all possible error patterns."

$$2^{n-k} \geq \sum_{i=0}^{t} {}^nC_i$$

→ Table look up decoding [Syndrome decoding]

Using the std array,

C_1 = all 0's	C_2	C_3	...	C_{2^k}
E_2	$C_2 + E_2$	$C_3 + E_2$	....	$C_{2^k} + E_2$
E_3	$C_2 + E_3$	$C_3 + E_3$	...	$C_{2^k} + E_3$
⋮				⋮
$E_{2^{n-k}}$	$C_2 + E_{2^{n-k}}$	$C_3 + E_{2^{n-k}}$	......	$C_{2^k} + E_{2^{n-k}}$

Let us suppose that $C_i = C_1, C_2, C_3 \ldots, C_{2^k}$ be the 2^k distinct code vectors of (n, k) LBC. Let E be any error pattern. Then the 2^k distinct error patterns E_i corresponding to the error E is given by:

$E_i = E + C_i$

The set of error vectors E_i is called COSET of the code. A COSET contains exactly 2^k elements that differ atmost by a code vector ie for (n, k) LBC, there are 2^{n-k}

COSETs.

$$E_i = E + C_i$$

$$E_i H^T = EH^T + \cancel{C_i H^T}^{\,0}$$

$$\Rightarrow E_i H^T = EH^T$$

From the above Eq^n it can be seen that It is independent of i.

All error patterns that differ atmost by a code vector have the same syndrome ie each COSET is characterised by an unique syndrome.

Standard array table Procedure:

Step ① The 2^k valid code vectors are placed in a row with all 0 code vector as the 1st element.

Step ② From the remaining $(2^n - 2^k)$ n tuples an n tuple E_2 is chosen & is placed below the all zero code vector. The 2nd row can now be formed by placing $E_2 + C_i$ where $i = 1, 2 \ldots 2^k$.

Step ③ Now an unused E_3 is taken & the 3rd row is completed. The process is continued till all n tuples are used.

→ Properties of Standard Array

1) Each element in the Std array is distinct & Hence diff columns of the array are disjoint.

2> The 1^{st} n tuple of each COSET is called COSET leader. If the error pattern caused by the channel coincides with a COSET leader; then the received vector is correctly decoded. If the error pattern is not a COSET leader then incorrect decoding results. Thus COSET leaders are ~~from~~ called correctable error patterns. (41)

From property 2: we conclude that when forming an array, Error patterns of a smallest weight should be chosen as coset leader. When this condn satisfies, then the decoding based on Std array would be the min distn decoding or max ~~liffely~~ know likely decoding.

3> All the 2^k ~~on~~ ntuple of a COSET have the same syndrome & the syndromes of different COSETs differ.

Decoding Procedure using Std Array

1> $S = RH^T$

2> The COSET leader E_i is located which has the syndrome S, E_i will be the error pattern.

3> The corrected vector C_i is obtained from $C_i = R + E_i$.

The above scheme will correct 2^{n-k} most likely error patterns introduced by the channel.

① Construct the std array for (6, 3) Code whose code vectors are (000000), (001110) (010011), (011101), (100101), (101011) (110110) and (111000). [std array prob.]

Solⁿ

Syndrome H^T	Co-set leader				
000	000000	00[illegible]0	010011	011101	100101
101	100000	[illegible]01110	110011	111101	000101
011	010000	01[illegible]10	000011	001101	110101
110	001000	0 001 [illegible]0	011011	010101	101101
100	000100	00[illegible]0[illegible]0	010111	011001	10000
010	000010	001[illegible]00	010001	01111[illegible]	100111
001	000001	00[illegible]11[illegible]	010010	011100	10010
111	110000	111[illegible]0	100011	101101	01010

Decoding using Std array

Let $R = [1\,0\,0\,1\,0\,0]$

$S = RH^T$

$$S = [1\,0\,0\,1\,0\,0] \begin{bmatrix} \cancel{1\,0\,0} \\ 1\;0\;1 \\ 0\;1\;1 \\ 1\;1\;0 \\ 1\;0\;0 \\ 0\;1\;0 \\ 0\;0\;1 \end{bmatrix} = [0\;0\;1]$$

$[0\,1\,0\,0\,1\,0]$ $\quad$ $[0\,0\,1]$

$[0\,1\,0\,0\,1\,1]$

Syndrome $S = [0\,0\,1]$ which is present in 6th row of H^T which means the 6th bit in the received vector is in error. [or 7th row in std array]

{∴ changing the 6th bit from 0 to 1, we have $C = [1\,0\,0\,1\,0\,1]$} normal method

For this co-set leader is 000001, this is the correctable error pattern.

$E = 000001$

$C = R + E = [100101]$

In the table $R = [100100]$ is located in 6th column & 7th row.

101011	110110	111000	
001011	010110	011000	
111011	100110	101000	
100011	111110	110000	
101111	110010	111100	
101001	110100	111010	
101010	110111	111001	
011011	000110	001000	

∴ The corrected code vector lies on top of 6th column namely [100101]

Let $R = [000011]$

$S = RH^T$

$$= [000011] \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [011]$$

Syndrome $S = [011]$ which is present in 2nd row of H^T or 3rd row of syndrome, which is 2nd bit in the received vector is in error. For this co-set leader is 010000, this is the correctable error pattern.

$C = [010011]$

→ For the systematic (7,4) LBC, the valid code vectors are (0000000), (0001011), (0010101), (0011110), (0100110), (0101101) (0110011) (0111000), (1000111), (1001100), (1010010), (1011001), (1100001), (1101010) (1110100) & (1111111).

Construct the std array for the code & Express the non zero component of the co-set leader in terms of syndrome bits S_1, S_2 & S_3.

→ <u>General Decoding Circuit</u> → PTO

<u>Soln</u> $(n,k) = (7,4)$

$$[H]_{(n-k)\times n} \Rightarrow H^T = [H^T]_{n\times(n-k)} = [H^T]_{7\times 3}$$

~~Let R = [1000100]~~

$$\cancel{S = RH^T = [1000100]} \cancel{\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [0\,0\,0]}$$

$\cancel{E = [0000010]}$

$\cancel{R = [1000100]}$

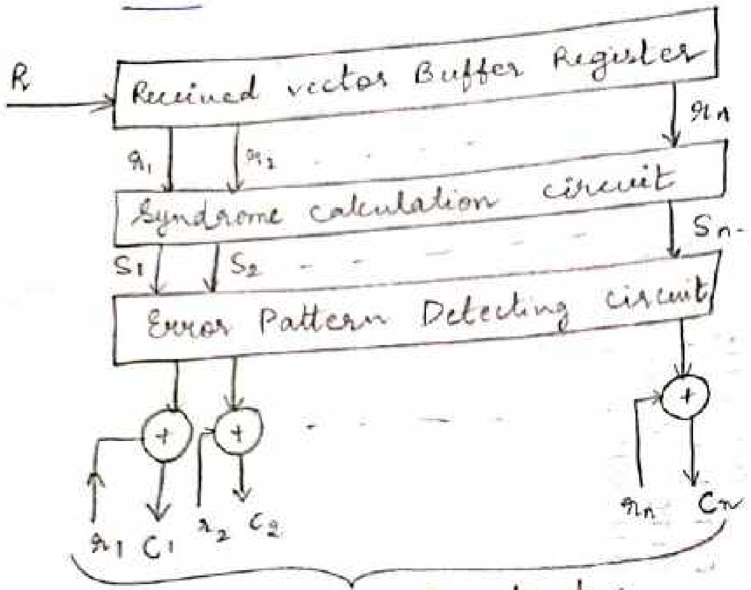
$\cancel{C = R+E = [1000110]}$

Let $R = [0010100]$

Syndrome	Co-set Leader								11-A	
000	0000000	0001011	0010101	0011110	0100110	0101101	0110011	0111000	1000111	1001100	1010010
110	1000000	1001011	1010101	1011110	1100110	1101101	1110011	1111000	0000111	0001100	0010010
101	0100000	0101011	0110101	0111110	0000110	0001101	0010011	0011000	1100111	1101100	1110010
011	0010000	0011011	0000101	0001110	0110110	0111101	0100011	0101000	1010111	1011100	1000010
111	0001000	0000011	0011101	0010110	0101110	0100101	0111011	0110000	1001111	1000100	1011010
100	0000100	0001111	0010001	0011010	0100010	0101001	0110111	0111100	1000011	1001000	1010110
010	0000010	0001001	0010111	0011100	0100100	0101111	0110001	0111010	1000101	1001110	1010000
001	0000001	0001010	0010100	0011111	0100111	0101100	0110010	0111001	1000110	1001101	1010011

	1011001	1100001	1101010	1110100	1111111
	0011001	0100001	0101010	0110100	0111111
	1111001	1000001	1001010	1010100	1011111
	1001001	1110001	1111010	1100100	1101111
	1010001	1101001	1100010	1111100	1110111
	1011101	1100101	1101110	1110000	1111011
	1011011	1100011	1101000	1110110	1111101
	1011000	1100000	1101011	1110101	1111110

→ <u>General Decoding circuit for (n, k) LBC</u>



corrected output

For the implementation decoding ckt, the std array can be regarded as in the truth table of 'n' switching func^ns (consisting of only 2 columns namely syndrome & co-set leader).

$e_1 = f_1(s_1, s_2, \ldots s_{n-k})$

$e_2 = f_2(s_1, s_2, \ldots s_{n-k})$

$\vdots$

$e_n = f_n(s_1, s_2 \ldots s_{n-k})$

where $s_1, s_2 \ldots s_{n-k}$ → are the syndrome digits are considered as switching variables & $e_1, e_2 \ldots, e_n$ → are the estimated error digits of different co-set leaders.

→ For (6, 3) LBC complete error correc...

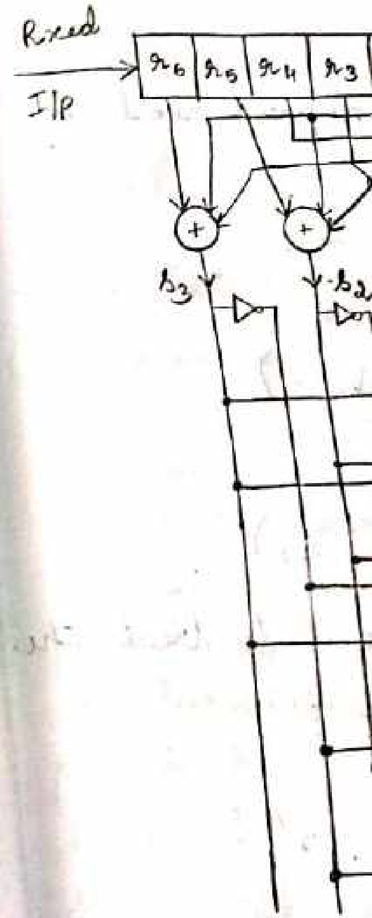
Refer Page 33.

$S = [r_1 + r_3 + r_4 \ldots$

From the std arr...

$e_1 = s_1 \bar{s}_2 s_3, \ldots$

$e_4 = s_1 \bar{s}_2 \bar{s}_3$, $e_5 \ldots$



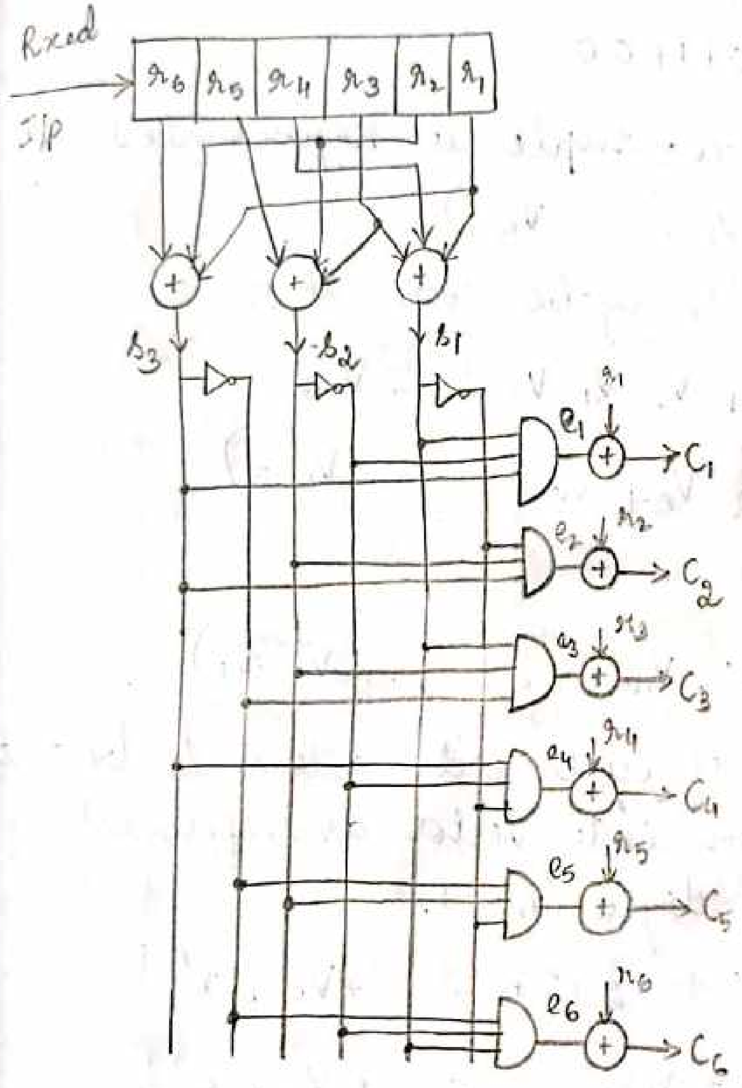
→ For (6,3) LBC in prob 1, Draw the complete error correcting ckt. (44)

Refer Page 33.

$$S = [r_1 + r_3 + r_4 \; , \; r_2 + r_3 + r_5 \; , \; r_1 + r_2 + r_6] \longrightarrow (1)$$

From the std array table, we have

$e_1 = s_1 \bar{s}_2 s_3$, $e_2 = \bar{s}_1 s_2 s_3$, $e_3 = s_1 s_2 \bar{s}_3$,
$e_4 = s_1 \bar{s}_2 \bar{s}_3$, $e_5 = \bar{s}_1 s_2 \bar{s}_3$, $e_6 = \bar{s}_1 \bar{s}_2 s_3$



10|11|17

Binary Cyclic Codes

→ **Algebraic Structure of Cyclic code**

Def^n → A (n, k) LBC 'C' is said to be cyclic if every cyclic shift of the code is also a code vector.

Ex : $C_1 = 0111001$

$C_2 = 1011100$

In general, n-tuple is represented by

$V = (V_0, V_1, V_2 \ldots V_{n-1})$

If V belongs to cyclic code then

$V^{(1)} = (V_{n-1}\ V_0\ V_1\ V_2 \ldots, V_{n-2})$

$V^{(2)} = (V_{n-2}\ V_{n-1}\ V_0\ V_1 \ldots V_{n-3})$

$\vdots$

$V^{(i)} = (V_{n-i}, V_{n-i+1}, \ldots V_{n-i-1})$

This property of cyclic code allows to treat the elements of each code vector as coefficients of polynomial of degree $(n-1)$.

$V(x) = V_0 + V_1 x + V_2 x^2 + \ldots + V_{n-1} x^{n-1}$

$V^{(1)}(x) = V_{n-1} + V_0 x + V_1 x^2 + \ldots + V_{n-2} x^{n-1}$

★ $1+1=0$

$x+x=0$

$x^3+x^3=0$

$x \cdot x = x^2$

$0+1=1$

$1+0=1$

→ Find the produc

1) $f_1(X) = 1+X+X$

$f_2(X) = 1+X+X$

soln $f_1(x) \cdot f_2(x)$

$f_1(x)\ f_2(x) = 1+X$

X^5+

$f_1(x)\ f_2(X) =$

2) $f(X) = (X+1)($

$= X^4+X$

$f(X) = 1+X^2$

→ DEFINI

(i) Galois fiel

Consider

to add^n &

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Fields with

primitive po

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polynomial o

using a ne

such that

When $P(X)$

a primitiv

Ex: X^3+

→ Pro

(a) For (n

-tor p

→ Find the product of polynomials :-

1) $f_1(X) = 1 + X + X^3$
$f_2(X) = 1 + X + X^2 + X^4$

soln $f_1(X) \cdot f_2(X) = (1 + X + X^3)(1 + X + X^2 + X^4)$

$f_1(X)\, f_2(X) = 1 + \cancel{X} + \cancel{X^2} + \cancel{X^4} + \cancel{X} + \cancel{X^2} + \cancel{X^3} + \cancel{X^5} + \cancel{X^3} + \cancel{X^4} + \cancel{X^5} + X^7$

$f_1(X)\, f_2(X) = 1 + X^7$

---

2) $f(X) = (X+1)(X^3 + X + 1)$.
$= X^4 + X^2 + \cancel{X} + X^3 + \cancel{X} + 1$
$f(X) = 1 + X^2 + X^3 + X^4$

→ DEFINITIONS:

(i) Galois field

Consider the set (0, 1) together with modulo to add^n & mul^n. Such a field is usually called a binary field or GF(2).

Fields with 2^m symbols are referred to as primitive polynomial.

The algebra of GF(2^m) can be derived using a polynomial of degree 'm' & with binary coefficients using a new variable 'α' called primitive element such that $P(\alpha) = 0$.

When $P(X)$ is irreducible then it is said to be a primitive polynomial.

Ex: $X^3 + X^2 + 1$, $X^3 + X + 1$, $X^5 + X^2 + 1$.

→ Properties of Cyclic codes.

(a) For (n, K) cyclic code, there exists a generator polynomial of degree (n, K) given by

$g(X) = g_0 + g_1 X + g_2 X^2 + \dots + g_{n-k} X^{n-k}$

(b) The generator polynomial of $g(X)$ is a factor of $X^n + 1$ such that $X^n + 1 = g(X) h(X)$

$h(X) \rightarrow$ parity check polynomial.

(c) If $g(X)$ is a polynomial of degree (n, k) & is a factor of $X^n + 1$, then it generates (n, k) cyclic codes.

(d) The code vector polynomial is given by

$$V(X) = D(X) \cdot g(X)$$

$$D(X) = d_0 + d_1 X + \dots + d_{k-1} X^{k-1}$$

This method generates non systematic cyclic codes.

(e) To generate a systematic cyclic code, the remainder polynomial $R(X)$ is obtained from division of $X^{n-k} D(X)$ by $g(X)$.

n bit code vector	
Coefficients of $R(X)$	coefficients of $D(X)$

$$R(X) = \frac{X^{n-k} D(X)}{g(X)}$$

$\longrightarrow$ For $(7,4)$ single error correcting cyclic code $D(X) = d_0 + d_1 X + d_2 X^2 + d_3 X^3$ &

$\longrightarrow (X^n + 1) = X^7 + 1 = (1 + X + X^3)(1 + X + X^2 + X^4)$

using the generator polynomial, $g(X) = 1 + X + X^3$

Find cyclic code in both non systematic & systematic form for $D = 1001$ & for $D = 1101$

Soln $D = 1001 = (d_0\ d_1\ d_2\ d_3)$ (46)

$D(X) = 1 + X^3$, $g(X) = 1 + X + X^3$

$V(X) = D(X) \cdot g(X)$

$= (1+X^3)(1+X+X^3)$

$= 1 + X + X^3 + X^3 + X^4 + X^6$

$V(X) = 1 + X + X^4 + X^6$

$= [1100101]$

For $D = 1101$

$V(X) = (1+X+X^3)(1+X+X^3)$

$= 1 + \cancel{X} + \cancel{X^3} + \cancel{X} + X^2 + \cancel{X^4} + \cancel{X^3} + \cancel{X^4} + X^6$

$= 1 + X^2 + X^6$

$V(X) = [1010001]$

Systematic Form

$R(X) = \dfrac{X^{n-k} D(X)}{g(X)} = \dfrac{X^3(1+X^3)}{1+X+X^3}$

$R(X) = \dfrac{X^3 + X^6}{1+X+X^3}$

$$\begin{array}{r} X^3 + X \\ X^3+X+1 \overline{\smash{)}\, \cancel{X^6} + X^3} \\ \cancel{X^6} + X^4 + X^3 \\ \hline \cancel{X^4} \\ \cancel{X^4} + X^2 + X \\ \hline X^2 + X \end{array}$$

$R(X) = X + X^2$

$R(X) = [011]$

$V = [011 \vdots 1001]$

(ii) $R(X) = \dfrac{X^3(1+X+X^3)}{1+X+X^3}$

$$\begin{array}{r} X^3 \\ X^3+X+1 \overline{\smash{)}\, X^6 + X^4 + X^3} \\ X^6 + X^4 + X^3 \\ \hline 0 \end{array}$$

$R(X) = [000]$

$V = [000 \vdots 1101]$

Scanned with CamScanner

→ Generation of Generator matrix & Parity check matrix

For (7, 4) code given $g(X) = 1 + X + X^3$
Find G & H matrices.

Soln $g(X) = 1 + X + X^3 = 1101000$

Since it is (7, 4) code, the order of G matrix is $K \times n$ ie 4×7

$$X\,g(X) = X + X^2 + X^4 = 0110100$$

$$X^2\,g(X) = X^2 + X^3 + X^5 = 0011010$$

$$X^3\,g(X) = X^3 + X^4 + X^6 = 0001101$$

$$G = \left[\begin{array}{ccc|cccc} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]_{4\times 7}$$

$r_3 \rightarrow r_3 + r_1$

$r_4 \rightarrow r_4 + r_1 + r_2$

$$G = \left[\begin{array}{ccc|cccc} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

To find H matrix

$$X^n + 1 = g(X) \cdot h(X)$$

$$h(X) = \frac{X^n + 1}{g(X)} = \frac{X^7 + 1}{1 + X + X^3}$$

$$= X^4 + X^2 + X + 1$$

$$h(x) = 1 + x + x^2 + x^4$$

$X^3 + X + 1 \,)\, X^7 + 1 \,(\, X^4 + X^2 \ldots$
$\quad \cancel{X^7} + X^5 + X^4$
$\quad \cancel{X^5} + X^4 + 1$
$\quad \cancel{X^5} + X^3 + X^2$
$\quad \cancel{X^4} + X^3 + \cancel{X^2} + 1$
$\quad \cancel{X^4} + \cancel{X^2} + X$

$\cancel{h(X) = 1110100}$

The order of H matrix is $(n-k) \times n = 3 \times 7$

$h(X^{-1}) = 1 + \frac{1}{X} + \frac{1}{X^2} + \frac{1}{X^4}$ || $X^k h(X^{-1})$

$X^4 h(X^{-1}) = X^4 + X^3 + X^2 + 1 = 1011100$

$X^5 h(X^{-1}) = X^5 + X^4 + X^3 + X = 0101110$

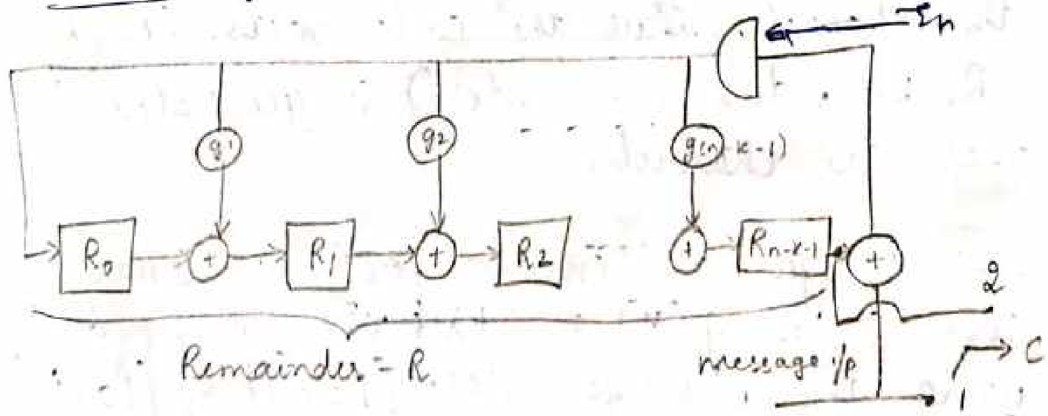
$X^6 h(X^{-1}) = X^6 + X^5 + X^4 + X^2 = 0010111$

$$H = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$r_1 \rightarrow r_1 + r_3$

$$H = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

→ Encoding using $(n-k)$ bit Shift Register



$\boxed{R}$ – FFs useds as shift regs. $(d_0 d_1 d_2 \ldots d_{k-1})$

(g_i) – closed if $g_i = 1$ & open if $g_i = 0$

$(+)$ – modulo - 2 adders.

Operation : In order to obtain the remainder polynomial $R(X)$, we need to perform division of $X^{n-k} \cdot D(X)$ by generator polynomial $g(X)$.

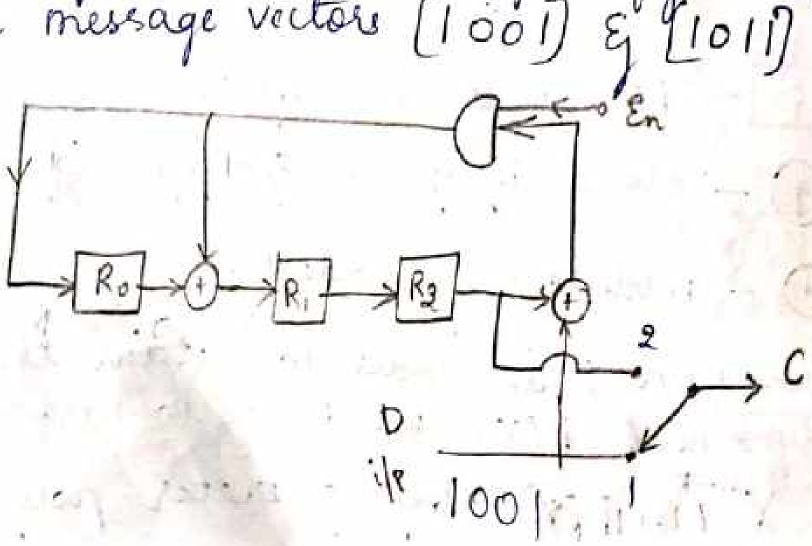
This division can be accomplished using the circuitary shown, which consists of feedback shift regs.

It is assumed that @ every clk pulse, the register i/ps are shifted into the register & appear at the o/p at the end of clk pulse.

1> With the gate turned on ie. enable = 1 & switch in posn ①, the info digits d_0, d_1 ---- d_{k-1} are shifted into the register with d_{k-1} first & simultaneously into the communication channel. As soon as the k info digits have been shifted into the register, the reg contains the parity check bits R_0, R_1 & ... R_{n-k-1}.

2> With the gate turned off, switch in posn ②, the contents of shift register are shifted into the channel. Thus the code vector ($R_0, R_1, ---, R_{n-k-1}, d_0, d_1, ---, d_{k-1}$) is generated & sent over the channel.

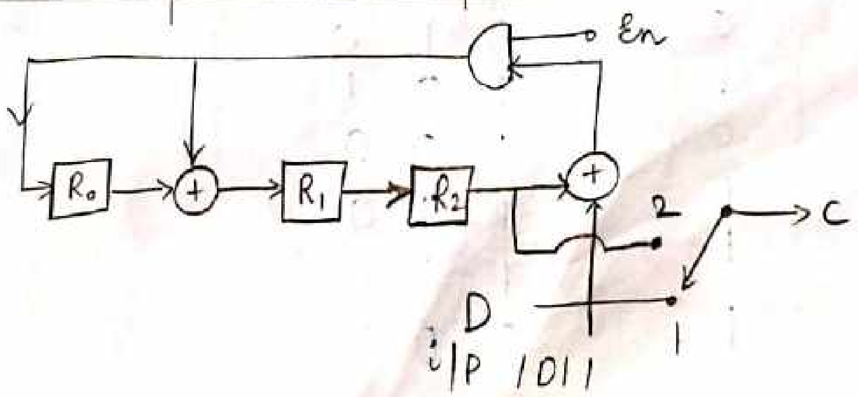
→ Design an encoder for (7,4) binary cyclic code given $g(X) = 1 + X + X^3$ & verify its opern using the message vectors [1001] & [1011]



No of shifts	I/P D	Shift Register R_0 R_1 R_2	Remainder bit R
Initialization switch in pos^n=1			& Enable =1
		0 0 0	—
1	1	1 1 0	
2	0	0 1 1	
3	0	1 1 1	
4	1	0 1 1	
switch in	position 2	& enable =0	
5	X	0 0 1	1
6	X	0 0 0	1
7	X	0 0 0	0

$V = [0 1 1 \vdots 1 0 0 1]$

No of shifts	I/P D	Shift Reg R_0 R_1 R_2	Remainder bit R
Initialization switch in pos^n=1		0 0 0	& enable =1 —
1	1	1 1 0	
2	1	1 0 1	
3	0	1 0 0	
4	1	1 0 0	
switch in	pos^n 2	& enable =0.	
5	x	0 1 0	0
6	x	0 0 1	0
7	x	0 0 0.	1

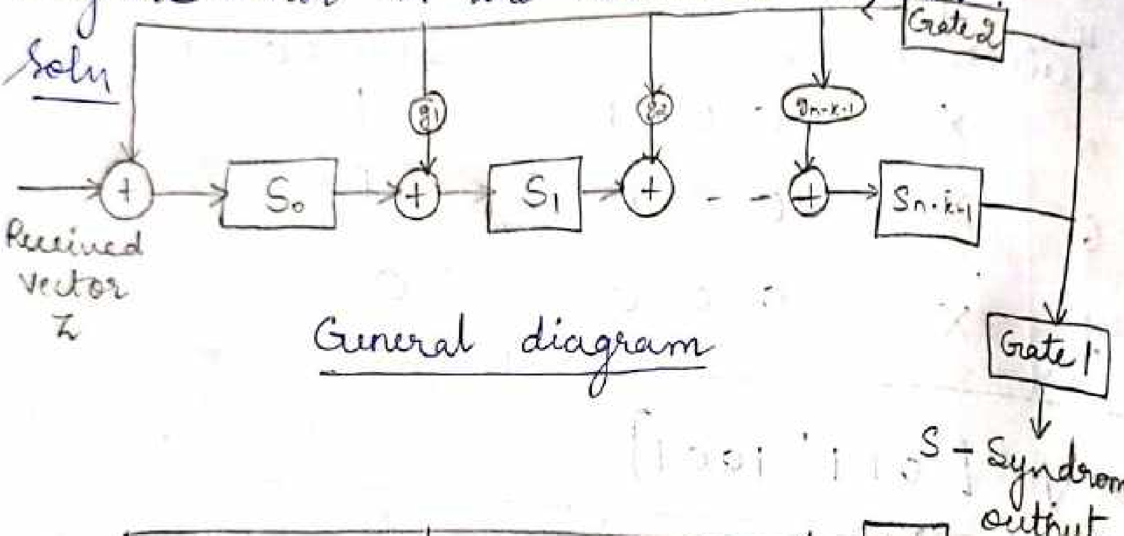


→ Syndrome Calculation

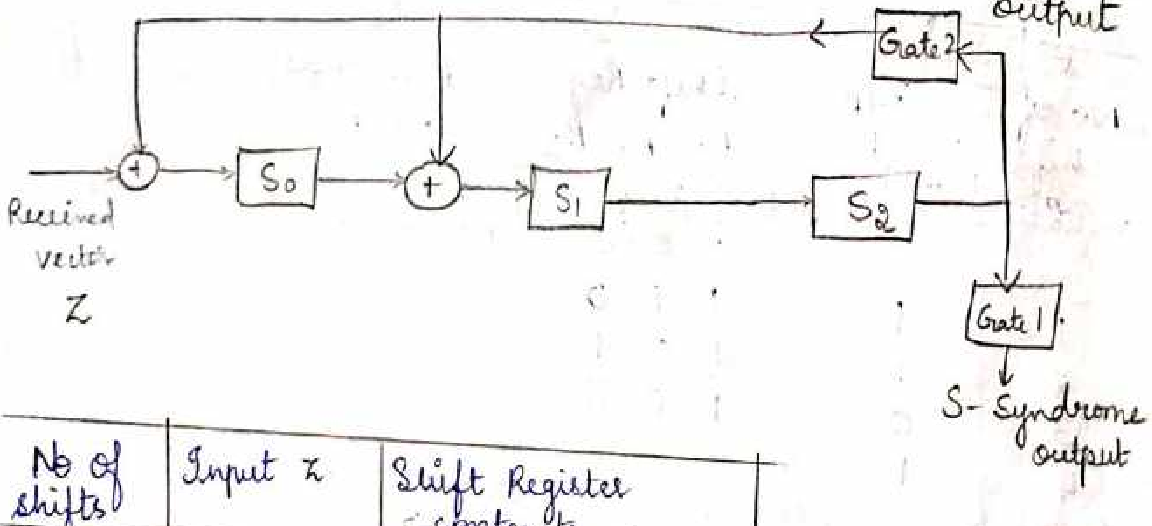
Error Detection & Correction

Problem: For (7,4) cyclic code, the received vector $z(s) = [1110101]$. The generator polynomial is $g(x) = 1 + X + X^3$. Draw the syndrome calculation circuit & correct the single error in the received vector.

Soln



General diagram



No of shifts	Input z	Shift Register contents S_0	S_1	S_2
Initialization Gate 1 off Gate 2 on		0	0	0
1	1	1	0	0
2	0	0	1	0
3	1	1	0	1
4	0	1	0	0
5	1	1	1	0
6	1	1	1	1
7	1	0	0	1

Gate 1 is enabled, Obtain the syndrome vector ie $S_0, S_1, S_2 = 001$ which is non zero. This indicates that there is an error

$$H^T = \begin{bmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \\ 1&1&0 \\ 0&1&1 \\ 1&1&1 \\ 1&0&1 \end{bmatrix}$$

Syndrome is 001, ie 3rd row of H^T, Hence the 3rd bit of z is to be changed

Hence [1100101] is the answer.

→ Repeat the same for received vector [0100101]

Soln

No of shifts	Input z	Shift Register contents S_0	S_1	S_2
		0	0	0
Initialization	Gate 1 off.	Gate 2 on		
1	1	1	0	0
2	0	0	1	0
3	1	1	0	1
4	0	1	0	0
5	0	0	1	0
6	1	1	0	1
7	0	1	0	0

Gate 1 is enabled, Obtain the syndrome vector ie $S_0 S_1 S_2 = 100$ which is non zero. This indicates that there is an error in syndrome which is the 1st row of H^T Hence 1st bit of Received vector must be changed.

[1100101]

Module 5

→ Convolution Codes

(n, k, m)

n → no of outputs or no. of modulo 2 adder

k → no of input bits entering at any time

m → no of stages of shift register or no of flip flops.

L → no of bits in the message sequence

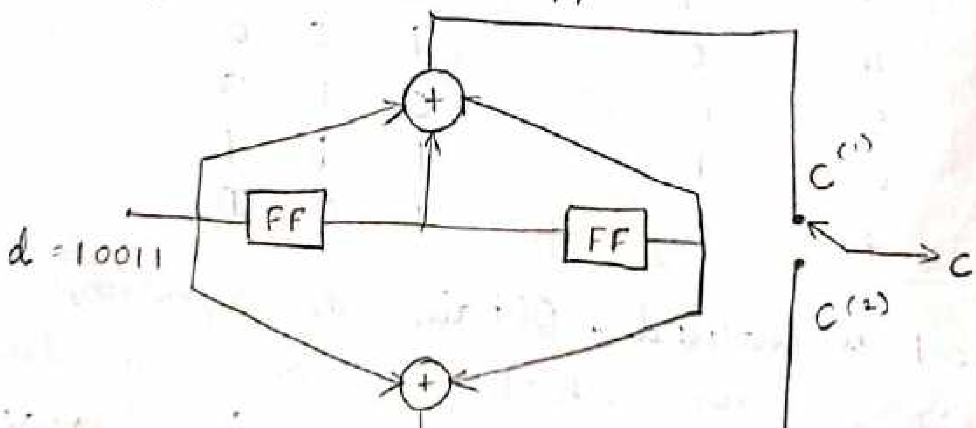
Constraint length = $m \times n$ digits.

Rate efficiency $= \frac{k}{n}$

→ For the convolution encoder shown in the figure, the info seq^n is $d = 10011$. Find the output sequence using

① Time domain approach

② Tranform domain approach



Soln ① Time domain approach

To find generator matrix, order is $L \times [n(L+m)]$

$n = 2$, $L = 5$, $m = 2$

$5[2(5+2)] = 5 \times 14$

From the 1st modulator we have,

$$\begin{array}{cccc} & g_1^{(1)} & g_2^{(1)} & g_3^{(1)} \\ g^{(1)} = & [\,1 & 1 & 1\,] \end{array}$$

$$\begin{array}{cccc} & g_1^{(2)} & g_2^{(2)} & g_3^{(2)} \\ g^{(2)} = & [\,1 & 0 & 1\,] \end{array}$$

$$G = \begin{bmatrix} 11 & 10 & 11 & 00 & 00 & 00 & 00 \\ 00 & 11 & 10 & 11 & 00 & 00 & 00 \\ 00 & 00 & 11 & 10 & 11 & 00 & 00 \\ 00 & 00 & 00 & 11 & 10 & 11 & 00 \\ 00 & 00 & 00 & 00 & 11 & 10 & 11 \end{bmatrix}_{5\times 14}$$

The output of the convolutional encoder is given by $[C] = [d][G]$

$$[C] = [1\,0\,0\,1\,1] \begin{bmatrix} 11 & 10 & 11 & 00 & 00 & 00 & 00 \\ 00 & 11 & 10 & 11 & 00 & 00 & 00 \\ 00 & 00 & 11 & 10 & 11 & 00 & 00 \\ 00 & 00 & 00 & 11 & 10 & 11 & 00 \\ 00 & 00 & 00 & 00 & 11 & 10 & 11 \end{bmatrix}$$

$$[C] = [11, 10, 11, 11, 01, 01, 11]$$

② Transform domain approach.

WKT $d = [10011]$

$d(x) = 1 + x^3 + x^4$

$g^{(1)}(x) = 1 + x + x^2$

$g^{(2)}(x) = 1 + x^2$

$C^{(i)}(x) = d(x)\, g^{(i)}(x)$

$c^{(1)}(x) = d(x)\, g^{(1)}(x)$

$= (1 + x^3 + x^4)(1 + x + x^2)$

$= 1 + x + x^2 + x^3 + \cancel{x^4} + \cancel{x^5} + \cancel{x^4} + \cancel{x^5} + x^6$

$= 1 + x + x^2 + x^3 + x^6 = [1111001]$

$c^{(2)}(x) = (1 + x^3 + x^4)(1 + x^2) = 1 + x^2 + x^3 + x^5 + x^4 + x^6$

$= [1011111]$

In general, $C(x) = C^{(1)}(x^n) + xC^{(2)}(x^n) + x^2 C^{(3)}(x^n) + \ldots + x^{n-1} C^{(n)}(x^n)$.

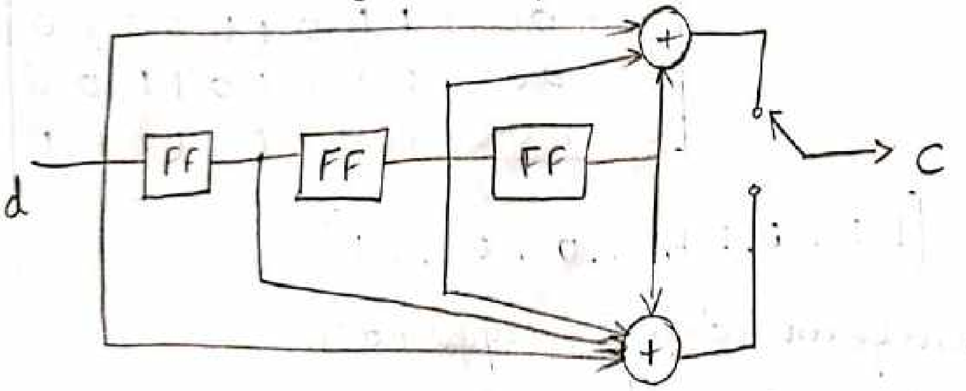
$$C(x) = C^{(1)}(x^2) + xC^{(2)}(x^2)$$

$$= 1 + x^2 + x^4 + x^6 + x^{12} + x + x^5 + x^7 + x^9 + x^{11} + x^{13}$$

$$C(x) = 1 + x + x^2 + x^4 + x^5 + x^6 + x^7 + x^9 + x^{11} + x^{12} + x^{13}$$

$$[C]_8 = [11, 10, 11, 11, 01, 01, 11]$$

→ Consider (2, 1, 3) convolutional encoder. Find the code word for $d = [10111]$: Apply Time domain & transform domain approach



Soln Time domain approach

To find generator matrix, order is $L \times (n(L+m))$

$$= 5 \times (2(5+3)) = 5 \times 16$$

From the 1st modulator, we have

$$g^{(1)} = [1011]$$

$$g^{(2)} = [1111]$$

$$G = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

the o/p of the convolutional encoder is given by $[c] = [d][G]$

$$[c] = [10111]\begin{bmatrix} 1&1&0&1&1&1&1&1&0&0&0&0&0&0&0&0 \\ 0&0&1&1&0&1&1&1&1&1&0&0&0&0&0&0 \\ 0&0&0&0&1&1&0&1&1&1&1&1&0&0&0&0 \\ 0&0&0&0&0&0&1&1&0&1&1&1&1&1&0&0 \\ 0&0&0&0&0&0&0&0&1&1&0&1&1&1&1&1 \end{bmatrix}$$

$$[c] = [11, 01, 00, 01, 01, 01, 00, 11]$$

(ii) Transform domain approach

WKT $d = [10111]$

$$d(x) = 1 + x^2 + x^3 + x^4$$

$$g^1(x) = 1 + x^2 + x^3$$

$$g^2(x) = 1 + x + x^2 + x^3$$

$$c^{(i)}(x) = d(x)\, g^{(i)}(x)$$

$$c^{(1)}(x) = (1 + x^2 + x^3 + x^4)(1 + x^2 + x^3)$$

$$= 1 + \cancel{x^2} + \cancel{x^3} + \cancel{x^2} + \cancel{x^4} + \cancel{x^5} + \cancel{x^3} + \cancel{x^5} + \cancel{x^6} + \cancel{x^4} + \cancel{x^6} + x^7$$

$$c^{(1)}(x) = 1 + x^7$$

$$c^{(2)}(x) = (1 + x^2 + x^3 + x^4)(1 + x + x^2 + x^3)$$

$$= 1 + x + \cancel{x^2} + \cancel{x^3} + \cancel{x^2} + x^3 + \cancel{x^4} + \cancel{x^5} + x^3 + \cancel{x^4} + \cancel{x^5} + \cancel{x^6} + x^4 + x^5 + \cancel{x^6} + x^7$$

$$c^{(2)}(x) = 1 + x + x^3 + x^4 + x^5 + x^7$$

$$c(x) = c^{(1)}(x^2) + x\,c^{(2)}(x^2)$$

$$= 1 + x^{14} + x + x^3 + x^7 + x^9 + x^{11} + x^{15}$$

$$c(x) = 1 + x + x^3 + x^7 + x^9 + x^{11} + x^{14} + x^{15}$$

$$= [11, 01, 00, 01, 01, 01, 00, 11]$$

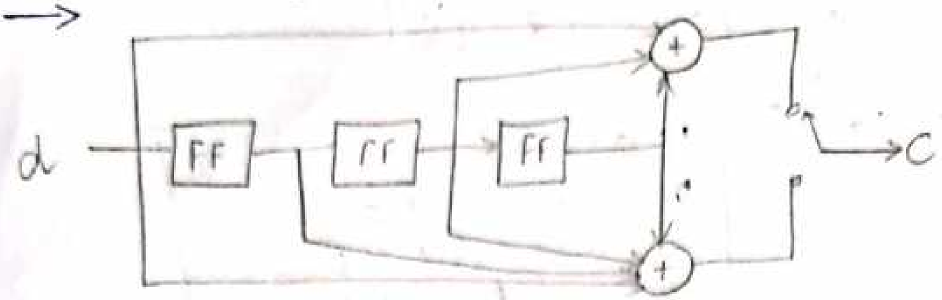
Formula:

$$c(x) = c^{(1)}(x^n) + x\,c^{(2)}(x^n) + x^2 c^{(3)}(x^n) + \cdots + x^{n-1} c^{(n)}(x^n)$$

14|11|17

(2, 1, 3) Convolutional encoder

③ →



Draw the state diagram, tree diagram, Trellis diagram. Find the code word for the message seq^n [10111] from the tree diagram.

Soln State Table

Present state (1 2 3)		Input (i)	Next state	output $c^{(1)}$	$c^{(2)}$
S_0	000	0	S_0	0	0
	000	1	S_1	1	1
S_1	100	0	S_2	0	1
	100	1	S_3	1	0
S_2	010	0	S_4	1	1
	010	1	S_5	0	0
S_3	110	0	S_6	1	0
	110	1	S_7	0	1
S_4	001	0	S_0	1	1
	001	1	S_1	0	0
S_5	101	0	S_2	1	0
	101	1	S_3	0	1
S_6	011	0	S_4	0	0
	011	1	S_5	1	1
S_7	111	0	S_6	0	1
	111	1	S_7	1	0

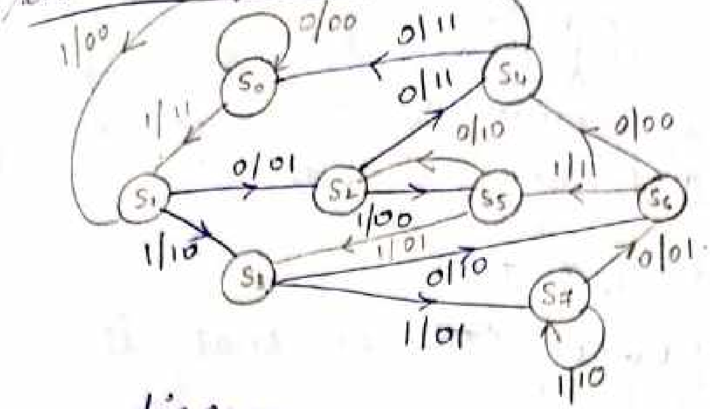
$g^{(1)} = [1011]$; $i+2+3 = c^{(1)}$

$g^{(2)} = [1111]$; $i+1+2+3 = c^{(2)}$

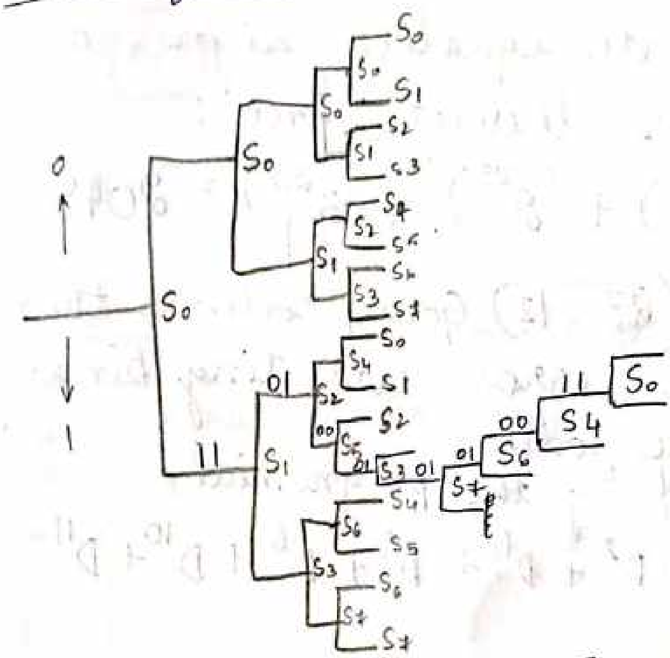
Code length $= n(L+m)$
$= 2(5+3) = 16$.

$(n, k, m) = (2, 1, 3)$.

State diagram : input / output



Tree diagram



[11, 01, 00, 01, 01, 01, 00, 11]

Golay Codes

- very special binary code that is capable of correcting any comb of 3 or fewer random errors in block of 23 bits.
- It has $d_{min} = 7$.
- It is a perfect code in that it satisfies Hamming bound.

for $t = 3$ with equality sign, as shown by number - theoretic fact:

$$1 + \binom{23}{1} + \binom{23}{2} + \binom{23}{3} = 2048 = 2^{11}$$

Indeed, the (23, 12) Golay code is the only known three - error correcting binary perfect cycle code.

(23,12) is generated by the polynomial.

$$g_1(D) = 1 + D^2 + D^4 + D^5 + D^6 + D^{10} + D^{11}$$

or by

$$g_2(D) = 1 + D + D^5 + D^6 + D^7 + D^9 + D^{11}$$

Both $g_1(D)$ & $g_2(D)$ are factors of $1+D^{23}$,

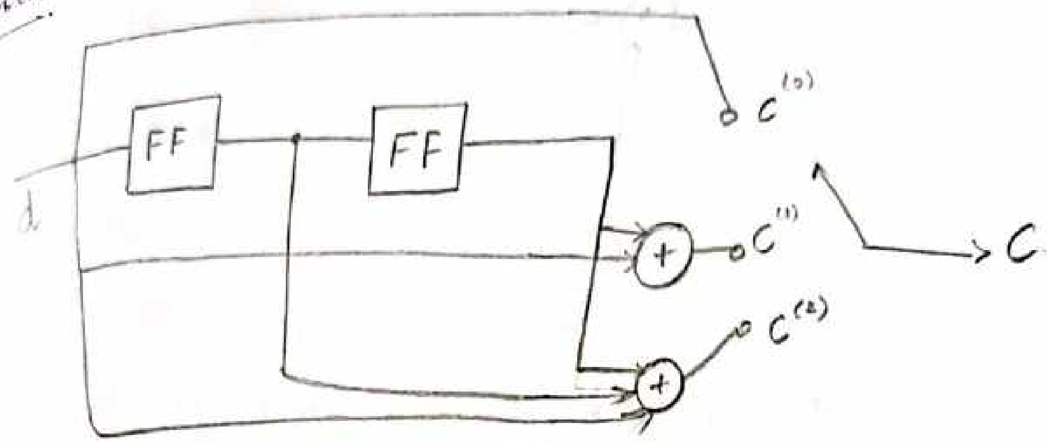
$$1 + D^{23} = (1+D)\, g_1(D) \cdot g_2(D).$$

It cannot be generalized for other n, k.

Scanned with CamScanner

→ For the Given ckt diagram (3, 1, 2), (53)
$D = [010110]$, $g^{(0)} = 100$, $g^{(1)} = 101$, $g^{(2)} = 111$
obtain the Trellis diagram for the received vector
[001, 110, 101, 110, 110, 010, 001, 100], Get the
estimated input.

Soln



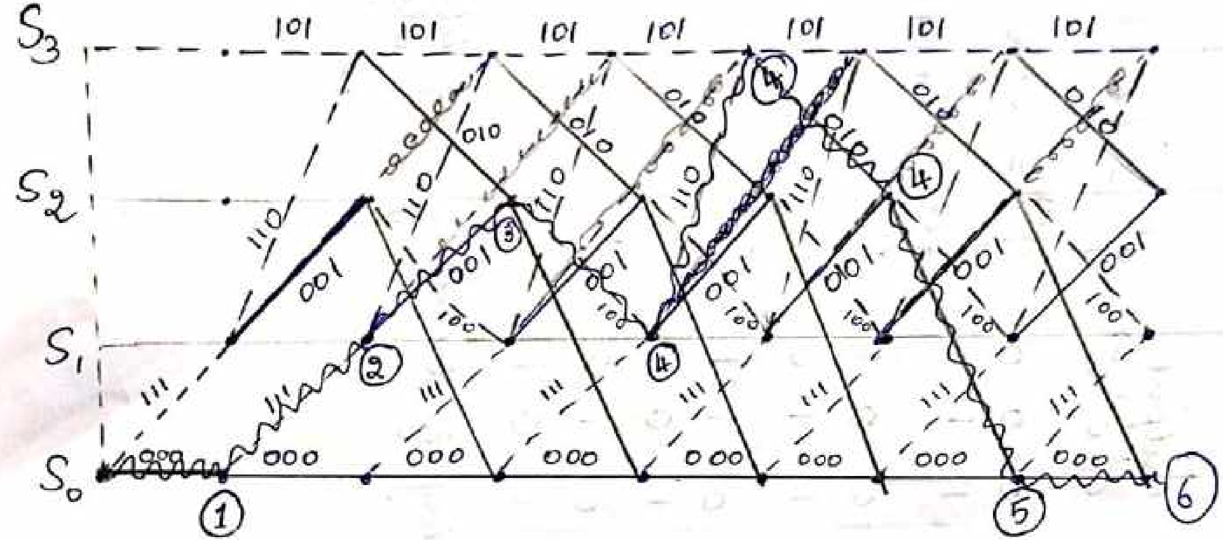
Present state 1 2	Input (i)	Next state	Output $c^{(0)}\ c^{(1)}\ c^{(2)}$
S_0 00	0	S_0	0 0 0
00	1	S_1	1 1 1
S_1 10	0	S_2	0 0 1
10	1	S_3	1 1 0
S_2 01	0	S_0	0 1 1
01	1	S_1	1 0 0
S_3 11	0	S_2	0 1 0
11	1	S_3	1 0 1

$g^{(0)} = i$

$g^{(1)} = i + 2$

$g^{(2)} = i + 1 + 2$

Trellis diagram [Encoding & decoding using Viterbi alg]



Received vector	001	110	101	110	110	010	001	100
Txed vector	000	111	001	100	110	010	011	000
Data	0	1	0	1	1	0	0	0

Base - Chaudhuri - Hocquenghem (BCH) code

- most imp & powerful classes of LBC.
- cyclic codes with a wide variety of parameters
- The most common BCH codes are characterized as follows :

for any +ve integer m (equal to or greater than 3) & t (less than $(2^m-1)/2$), there exists BCH code with parameters :-

Block length : $n = 2^m - 1$

no. of msg bits : $k \geq n - mt$

dmin : $d_{min} \geq 2t+1$

- t error correcting code.
- Hamming single error correcting codes can be described as BCH codes.
- offer flexibility in choice of code parameters namely block length (n) & code rate.